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Theoretical article

Euler's constant, generalized (II) - other ways of determining

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Abstract

In a previous paper, we generalized the well-known Euler-Mascheroni constant c_0, in both positive and negative senses. In this sense, for each $\alpha\in(0,1)$, starting from two generalizations of the string cn which converges to c_0, we obtained two sequences $c_{n,\alpha}$ and $c_{n,-\alpha}$, which converge to c_α and $c_{n,-\alpha}$, respectively. We called these limits, c_α - the positive generalized Euler-Mascheroni constant or the positive generalized Euler constant, respectively $c_{-\alpha}$ - the negative generalized Euler-Mascheroni constant or the negative generalized Euler constant. By calculating the limits of some sequences in two different ways, we obtained the integral form of these two constants c_α and $c_{-\alpha}$ and, then, we calculated these two constants for different rational values of the number α. In this paper we will present other ways of determining these generalized constants, mentioned above, and we have extended the determination of these generalized constants to different values of $\alpha\geq1$ (integers or rational numbers). With all the values obtained for c_α, we have presented immediate applications to determining the limits of sequences of real numbers. At the end of this paper, I proposed, to the attentive reader interested in these issues, the solution of an interesting exercise. Of course, this paper is exclusively about Mathematics Didactics and can be used / recommended to all those interested in these issues: pupils, students or Mathematics teachers.

Keywords:
limit, sequence, convergence, integral, Euler's constant.

1. Introduction

As part of a larger project aimed at "*Training and developing the skills of pupils, students and teachers to solve exercises and problems in Mathematics*", project started 20 years ago - see (Vălcan, 1994, 2016, 2017, 2018, 2021), we set out, in (Vălcan, 2024), to train and develop these competences for calculating the limits of certain types of competences. In this sense, we have generalized the well-known Euler-Mascheroni constant c_0, in positive and negative sense.

Thus, for each $\alpha\in(0,1)$, starting from two generalizations of the sequence c_n - which converges to c_0, we obtained two sequences $c_{n,\alpha}$ and $c_{n,-\alpha}$, which converge to c_α and $c_{-\alpha}$, respectively. We called these limits, c_α - the positive generalized Euler-Mascheroni constant or the positive generalized Euler constant, respectively $c_{-\alpha}$ - the negative generalized Euler-Mascheroni constant or the negative generalized Euler constant. By calculating the limits of some sequences in two different ways, we obtained the integral form of these two constants c_α and $c_{-\alpha}$ and, then, we calculated these two constants for different rational values of the number α.

All the works mentioned above, like this one, are from Didactics of Mathematics.

2. Theoretical foundation

Next, we present the main results obtained in (Vălcan, 2024).

To begin with, we presented and proved the following technical results, which we used further and which we list here, keeping the original numbering.

We considered the convergent sequence:

$$c_{n,0}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}+\frac{1}{n}-\ln n, \qquad (1.8)$$

and, for which:

$$\lim_{n\to\infty} c_{n,0}=\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}+\frac{1}{n}-\ln n\right)\overset{\text{not.}}{=} c_0. \qquad (1.9)$$

Then, we considered a generalized Euler-type sequence:

$$c_{n,\alpha}=\frac{1}{1+\alpha}+\frac{1}{2+\alpha}+\frac{1}{3+\alpha}+\ldots+\frac{1}{n+\alpha}-\ln(n+\alpha). \qquad (1.10)$$

We will show, further, that this sequence is convergent and:

$$\lim_{n\to\infty} c_{n,\alpha} = \lim_{n\to\infty}\left(\frac{1}{1+\alpha}+\frac{1}{2+\alpha}+\frac{1}{3+\alpha}+\dots+\frac{1}{n+\alpha}-\ln(n+\alpha)\right) \overset{not.}{=} c_\alpha. \quad (1.20)$$

Analogously, we would consider the sequence:

$$c_{n,-\alpha}=\frac{1}{1-\alpha}+\frac{1}{2-\alpha}+\frac{1}{3-\alpha}+\dots+\frac{1}{n-\alpha}-\ln(n-\alpha). \quad (1.21)$$

We have shown, further, that this sequence is also convergent and:

$$\lim_{n\to\infty} c_{n,-\alpha} = \lim_{n\to\infty}\left(\frac{1}{1-\alpha}+\frac{1}{2-\alpha}+\frac{1}{3-\alpha}+\dots+\frac{1}{n-\alpha}-\ln(n-\alpha)\right) \overset{not.}{=} c_{-\alpha}. \quad (1.31)$$

Then, we evaluated these constants c_α and $c_{-\alpha}$ and concluded that:

$$c_0-c_\alpha=\int_0^1 \frac{1-x^\alpha}{1-x}\cdot dx, \text{ that is: } c_\alpha=c_0-\int_0^1 \frac{1-x^\alpha}{1-x}\cdot dx, \quad (1.44)$$

and:

$$c_{-\alpha}-c_0=\int_0^1 \frac{1-x^\alpha}{x^\alpha\cdot(1-x)}\cdot dx, \text{ that is:}$$

$$c_{-\alpha}=c_0+\int_0^1 \frac{1-x^\alpha}{x^\alpha\cdot(1-x)}\cdot dx. \quad (1.50)$$

Because,

$$\lim_{\substack{x\to1\\x<1}} (1-x)^{1-\alpha}\cdot\frac{1-x^\alpha}{1-x}=0 \quad \text{and}$$

$$\lim_{\substack{x\to1\\x<1}} (1-x)^{1-\alpha}\cdot\frac{1-x^\alpha}{x^\alpha\cdot(1-x)}=0, \quad (1.51)$$

according to (Sirețchi, Corollary 3, p. 388), it follows that the integrals:

$$\int_0^1 \frac{1-x^\alpha}{1-x}\cdot dx \quad \text{and} \quad \int_0^1 \frac{1-x^\alpha}{x^\alpha\cdot(1-x)}\cdot dx$$

are convergent; therefore, all equalities in which they intervene are valid. Therefore, for any $\alpha\in(0,1)$, c_α and $c_{-\alpha}$ exist. We called these limits, c_α - the positive generalized Euler-Mascheroni constant or the positive generalized Euler constant, respectively $c_{-\alpha}$ - the negative generalized Euler-Mascheroni constant or the negative generalized Euler constant.

Next, we calculated several such Euler-type constants, generalized positively and negatively, respectively, for different values of $\alpha\in(0,1)$; thus we obtained:

1) For $\alpha=\frac{1}{2}$, according to the equalities from (1.44) and (1.50),

$$c_{\frac{1}{2}}=c_0-2+2\cdot\ln2 \quad \text{and} \quad c_{-\frac{1}{2}}=c_0+2\cdot\ln2. \quad (1.57)$$

2) For $\alpha=\frac{1}{3}$, according to the equalities from (1.44) and (1.50),

$$c_{\frac{1}{3}}=c_0-3+\frac{3}{2}\cdot\ln3+\frac{\pi\cdot\sqrt{3}}{6} \quad \text{and}$$

$$c_{-\frac{1}{3}}=c_0+\frac{3}{2}\cdot\ln3-\frac{\pi\cdot\sqrt{3}}{6}. \quad (1.61)$$

3) For $\alpha=\frac{2}{3}$, according to the equalities from (1.44) and (1.50),

$$c_{\frac{2}{3}}=c_0-\frac{3}{2}+\frac{3}{2}\cdot\ln3-\frac{\pi\cdot\sqrt{3}}{6} \quad \text{and}$$

$$c_{-\frac{2}{3}}=c_0+\frac{3}{2}\cdot\ln3+\frac{\pi\cdot\sqrt{3}}{6}. \quad (1.65)$$

At the end of the paper (Vălcan, 2024) I presented other expressions (including integrals) of the constants c_α and $c_{-\alpha}$ and I proposed to the attentive reader interested in these issues, the following:

1) The proof, according to the examples above, that:

$$c_{\frac{1}{4}}=c_0-4+3\cdot\ln2+\frac{\pi}{2},\ c_{-\frac{1}{4}}=c_0+3\cdot\ln2-\frac{\pi}{2}, \quad (1.75)$$

$$c_{\frac{3}{4}}=c_0+\frac{4}{3}+3\cdot\ln2+\frac{\pi}{2},\ c_{-\frac{3}{4}}=c_0+3\cdot\ln2+\frac{\pi}{2}. \quad (1.76)$$

Let it be calculated $c_{\frac{p}{5}}$, for $p\in\{-4,-3,-2,-1,1,2,3,4\}$.

3. Research methodology

In this paragraph, we will present other ways of determining the constants c_α and $c_{-\alpha}$ and, then, knowing their values, we will calculate, differently than we have done so far, the limits of some known strings. Therefore:

1) $\alpha=\frac{1}{2}$. In (Vălcan, 2024) we proved that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)}=1-\ln 2. \qquad (2.1)$$

But, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(2\cdot k)\cdot(2\cdot k+1)}=\frac{1}{2\cdot k}-\frac{1}{2\cdot k+1}.$$

So,

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)}=\sum_{k=1}^{n}\left(\frac{1}{2\cdot k}-\frac{1}{2\cdot k+1}\right)$$

$$=\frac{1}{2}\cdot\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{2\cdot k+1}=\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\left(\frac{1}{2\cdot 1+1}+\frac{1}{2\cdot 2+1}+\ldots+\frac{1}{2\cdot n+1}\right)$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\frac{1}{2}\cdot\left(\frac{1}{1+\frac{1}{2}}+\frac{1}{2+\frac{1}{2}}+\ldots+\frac{1}{n+\frac{1}{2}}\right)$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)-\frac{1}{2}\cdot\left(\frac{1}{1+\frac{1}{2}}+\frac{1}{2+\frac{1}{2}}+\ldots+\frac{1}{n+\frac{1}{2}}-\ln\left(n+\frac{1}{2}\right)\right)+\frac{1}{2}\cdot\ln n-\frac{1}{2}\cdot\ln\left(n+\frac{1}{2}\right)$$

$$=\frac{1}{2}\cdot\left(c_n-c_{n,\frac{1}{2}}\right)+\frac{1}{2}\cdot\ln\frac{2\cdot n}{2\cdot n+1}. \qquad (2.2)$$

Passing to the limit, when $n\to\infty$, from equalities (2.2), we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)}=\frac{1}{2}\cdot\left(c_0-c_{\frac{1}{2}}\right). \qquad (2.3)$$

Now, from equalities (2.1) and (2.3), we obtain the value of $c_{\frac{1}{2}}$ from (1.57).

Otherwise: We have the equalities:

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)}=\sum_{k=1}^{n}\left(\frac{1}{2\cdot k}-\frac{1}{2\cdot k+1}\right)$$

$$=\frac{1}{2}\cdot\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{2\cdot k+1}=\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\left(\frac{1}{2\cdot 1+1}+\frac{1}{2\cdot 2+1}+\ldots+\frac{1}{2\cdot n+1}\right)$$

$$=\frac{1}{2}\cdot c_n+\frac{1}{2}\cdot\ln n-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2\cdot n}+\frac{1}{2\cdot n+1}\right)+1+\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\frac{1}{2}\cdot c_n+\frac{1}{2}\cdot\ln n-c_{2\cdot n+1}-\ln(2\cdot n+1)+1+\frac{1}{2}\cdot c_n+\frac{1}{2}\cdot\ln n$$

$$=c_n-c_{2\cdot n+1}-\ln(2\cdot n+1)+1+\ln n$$

$$=1+c_n-c_{2\cdot n+1}+\ln\frac{n}{2\cdot n+1}. \qquad (2.4)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.4), we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)}=1-\ln 2, \qquad (2.5)$$

because the sequence $(c_n)_{n\geq 1}$ is convergent and, thus,

$$\lim_{n\to\infty}(c_n-c_{2\cdot n+1})=0.$$

From equalities (2.3) and (2.5), we again obtain the value of $c_{\frac{1}{2}}$ from (2.57).

Now, knowing the value of $c_{\frac{1}{2}}$, from the equalities in (2.57), we can calculate the limit of a sequence, otherwise. Thus, we have the equalities:

$$\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2}{2\cdot k+1}\right)$$

$$=\sum_{k=1}^{n}\frac{1}{k}-2\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k+1}=\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-2\cdot\left(\frac{1}{2\cdot 1+1}+\frac{1}{2\cdot 2+1}+\ldots+\frac{1}{2\cdot n+1}\right)$$

$$=c_n+\ln n-2\cdot\frac{1}{2}\cdot\left(\frac{1}{1+\frac{1}{2}}+\frac{1}{2+\frac{1}{2}}+\ldots+\frac{1}{n+\frac{1}{2}}\right)$$

$$=c_n+\ln n-c_{n,\frac{1}{2}}+\ln\frac{2}{2\cdot n+1}=c_n-c_{n,\frac{1}{2}}+\ln\frac{2\cdot n}{2\cdot n+1}. \quad (2.6)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.6), according to the first equality from (1.57), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k+1)}=2-2\cdot\ln 2. \quad (2.1')$$

2) $\alpha=-\frac{1}{2}$. First let us prove that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-1)}=\ln 2. \quad (2.7)$$

For every $k\in\mathbf{N}^*$,

$$\frac{1}{(2\cdot k)\cdot(2\cdot k-1)}=\frac{1}{2\cdot k-1}-\frac{1}{2\cdot k}=\int_0^1(x^{2\cdot k-2}-x^{2\cdot k-1})\cdot dx \quad (2.8)$$

So,

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-1)}=\sum_{k=1}^{n}\left(\frac{1}{2\cdot k-1}-\frac{1}{2\cdot k}\right)=\sum_{k=1}^{n}\int_0^1(x^{2\cdot k-2}-x^{2\cdot k-1})\cdot dx$$

$$=\sum_{k=1}^{n}\int_0^1 x^{2\cdot k-2}\cdot(1-x)\cdot dx=\int_0^1\sum_{k=1}^{n}x^{2\cdot k-2}\cdot(1-x)\cdot dx=\int_0^1(1-x)\cdot\sum_{k=1}^{n}x^{2\cdot k-2}\cdot dx$$

$$=\int_0^1(1-x)\cdot\frac{1-x^{2\cdot n}}{1-x^2}\cdot dx=\int_0^1\frac{1-x^{2\cdot n}}{1+x}\cdot dx=\int_0^1\frac{1}{1+x}\cdot dx-\int_0^1\frac{x^{2\cdot n}}{1+x}\cdot dx. \quad (2.9)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.9), according to (Vălcan, (I), 2016, Proposition), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-1)}=\int_0^1\frac{1}{1+x}\cdot dx=\ln 2.$$

Otherwise: We have the equalities:

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+1)}=\sum_{k=1}^{n}\left(\frac{1}{2\cdot k-1}-\frac{1}{2\cdot k}\right)=\sum_{k=1}^{n}\frac{1}{2\cdot k-1}-\frac{1}{2}\cdot\sum_{k=1}^{n}\frac{1}{k}=$$

$$\left(\frac{1}{2\cdot1-1}+\frac{1}{2\cdot2-1}+\ldots+\frac{1}{2\cdot n-1}\right)-\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\left(\frac{1}{1}+\frac{1}{3}+\ldots+\frac{1}{2\cdot n-1}\right)-\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2\cdot n-2}+\frac{1}{2\cdot n-1}+\frac{1}{2\cdot n}\right)-\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2\cdot n-2}+\frac{1}{2\cdot n-1}+\frac{1}{2\cdot n}-\ln(2\cdot n)\right)-\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)$$

$$+\ln(2\cdot n)-\ln n=c_{2\cdot n}-c_n+\ln 2. \quad (2.10)$$

Passing to the limit, when $n\to +\infty$, in equalities (2.10), we obtain equality (2.7), because the sequence $(c_n)_{n\geq1}$ is convergent and, thus,

$$\lim_{n\to\infty}(c_{2\cdot n}-c_n)=0.$$

On the other hand, from the equalities in (2.8), it follows that:

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-1)}=\sum_{k=1}^{n}\left(\frac{1}{2\cdot k-1}-\frac{1}{2\cdot k}\right)$$

$$=\sum_{k=1}^{n}\frac{1}{2\cdot k-1}-\frac{1}{2}\cdot\sum_{k=1}^{n}\frac{1}{k}=$$

$$\left(\frac{1}{2\cdot1-1}+\frac{1}{2\cdot2-1}+\ldots+\frac{1}{2\cdot n-1}\right)-\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}+\frac{1}{2-\frac{1}{2}}+\ldots+\frac{1}{n-\frac{1}{2}}\right)-\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}+\frac{1}{2-\frac{1}{2}}+\ldots+\frac{1}{n-\frac{1}{2}}-\ln\left(n-\frac{1}{2}\right)\right)-\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)+\frac{1}{2}\cdot\ln\left(n-\frac{1}{2}\right)-\frac{1}{2}\cdot\ln n=\frac{1}{2}\cdot(c_{n,-\frac{1}{2}}-c_n)+\frac{1}{2}\cdot\ln\frac{2\cdot n-1}{2\cdot n}. \quad (2.11)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.11), we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-1)}=\frac{1}{2}\cdot(c_{-\frac{1}{2}}-c_0). \quad (2.12)$$

Now, from equalities (2.7) and (2.12), we obtain the value of $c_{-\frac{1}{2}}$ from (1.57).

Finally, knowing the value of $c_{-\frac{1}{2}}$, from the equalities in (1.57), we can calculate the limit of a sequence, otherwise. Thus, we have the equalities:

$$\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k-1)}=\sum_{k=1}^{n}\left(\frac{2}{2\cdot k-1}-\frac{1}{k}\right)=2\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k-1}-\sum_{k=1}^{n}\frac{1}{k}=2\cdot\left(\frac{1}{2\cdot1-1}+\frac{1}{2\cdot2-1}+\ldots+\frac{1}{2\cdot n-1}\right)-\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)=$$

$$=2\cdot\frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}+\frac{1}{2-\frac{1}{2}}+\ldots+\frac{1}{n-\frac{1}{2}}\right)-c_n+\ln n=c_{n,-\frac{1}{2}}+\ln\frac{2}{2\cdot n-1}-c_n+\ln n=c_{n,-\frac{1}{2}}-c_n-+\ln\frac{2\cdot n}{2\cdot n-1}. \quad (2.13)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.13), according to the second equality from (1.57), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k-1)}=2\cdot\ln 2. \quad (2.7')$$

More than that, we have the equalities:

$$c_{n,\frac{1}{2}}=\frac{1}{1+\frac{1}{2}}+\frac{1}{2+\frac{1}{2}}+\ldots+\frac{1}{n+\frac{1}{2}}-\ln\left(n+\frac{1}{2}\right)=\frac{1}{1-\frac{1}{2}}+\frac{1}{2-\frac{1}{2}}+\ldots+\frac{1}{n-\frac{1}{2}}-\ln\left(n-\frac{1}{2}\right)+\frac{1}{n+\frac{1}{2}}-\ln\left(n+\frac{1}{2}\right)+\ln\left(n-\frac{1}{2}\right)-2$$

$$=c_{n,-\frac{1}{2}}+\frac{1}{n+\frac{1}{2}}+\ln\left(\frac{2\cdot n-1}{2\cdot n+1}\right)-2. \quad (2.14)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.14), we obtain that:

$$c_{\frac{1}{2}}=c_{-\frac{1}{2}}-2, \quad (2.15)$$

equality proven by equalities (1.57).

3) $\alpha=\frac{1}{3}$. In (Vălcan, 2024) we proved that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)}=1-\frac{1}{2}\cdot\ln 3-\frac{\pi\cdot\sqrt{3}}{18}. \quad (2.16)$$

But, for ev every $k\in\mathbf{N}^*$,

$$\frac{1}{(3\cdot k)\cdot(3\cdot k+1)}=\frac{1}{3\cdot k}-\frac{1}{3\cdot k+1}.$$

So,

$$\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)}=\sum_{k=1}^{n}\left(\frac{1}{3\cdot k}-\frac{1}{3\cdot k+1}\right)=\frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{3\cdot k+1}=\frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\left(\frac{1}{3\cdot1+1}+\frac{1}{3\cdot2+1}+\ldots+\frac{1}{3\cdot n+1}\right)$$

$$=\frac{1}{3}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\frac{1}{3}\cdot\left(\frac{1}{1+\frac{1}{3}}+\frac{1}{2+\frac{1}{3}}+\cdots+\frac{1}{n+\frac{1}{3}}\right)$$

$$= \frac{1}{3} \cdot \left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}-\ln n\right) - \frac{1}{3} \cdot$$

$$\left(\frac{1}{1+\frac{1}{3}}+\frac{1}{2+\frac{1}{3}}+...+\frac{1}{n+\frac{1}{3}}-\ln\left(n+\frac{1}{3}\right)\right)$$

$$+\frac{1}{3}\cdot\ln n-\frac{1}{3}\cdot\ln\left(n+\frac{1}{3}\right)=\frac{1}{3}\cdot(c_n+\ln n)-\frac{1}{3}\cdot$$

$$\left(c_{n,\frac{1}{3}}+\ln\left(n+\frac{1}{3}\right)\right)$$

$$=\frac{1}{3}\cdot(c_n-c_{n,\frac{1}{3}})+\frac{1}{3}\cdot\ln\left(\frac{3\cdot n}{3\cdot n+1}\right). \qquad (2.17)$$

Passing to the limit, when $n\to +\infty$, in equalities (2.17), we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+1)}=\frac{1}{3}\cdot(c_n-c_{n,\frac{1}{3}}). \qquad (2.18)$$

Now, from equalities (2.16) and (2.18), we obtain the value of $c_{\frac{1}{3}}$ from (1.61).

Finally, knowing the value of $c_{\frac{1}{3}}$, from the equalities in (1.61), we can calculate the limit of a string, otherwise. Thus, we have the equalities:

$$\sum_{k=1}^{n}\frac{1}{k\cdot(3\cdot k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{3}{3\cdot k+1}\right)$$

$$=\sum_{k=1}^{n}\frac{1}{k}-3\cdot\sum_{k=1}^{n}\frac{1}{3\cdot k+1}=\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}\right)-3\cdot$$

$$\left(\frac{1}{3\cdot 1+1}+\frac{1}{3\cdot 2+1}+...+\frac{1}{3\cdot n+1}\right)$$

$$=c_n+\ln n-3\cdot\frac{1}{3}\cdot\left(\frac{1}{1+\frac{1}{3}}+\frac{1}{2+\frac{1}{3}}+...+\frac{1}{n+\frac{1}{3}}\right)=c_n+\ln n-c_{n,\frac{1}{3}}+\ln\frac{3}{3\cdot n+1}$$

$$= c_n-c_{n,\frac{1}{3}}+\ln\frac{3\cdot n}{3\cdot n+1}. \qquad (2.19)$$

Passing to the limit, when $n\to +\infty$, in equalities (2.19), according to the first equality from (1.61), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(3\cdot k+1)}=3-\frac{3}{2}\cdot\ln 3-\frac{\pi\cdot\sqrt{3}}{6}. \qquad (1.16')$$

4) $\alpha=-\frac{1}{3}$. First let us prove that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k-1)}=\frac{1}{2}\cdot\ln 3-\frac{\pi\cdot\sqrt{3}}{18}. \qquad (2.20)$$

For every $k\in\mathbf{N}^*$,

$$\frac{1}{(3\cdot k)\cdot(3\cdot k+1)}=\frac{1}{3\cdot k-1}-\frac{1}{3\cdot k}=\int_0^1(x^{3\cdot k-2}-x^{3\cdot k-1})\cdot dx. \qquad (2.21)$$

So,

$$\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k-1)}=\sum_{k=1}^{n}\left(\frac{1}{3\cdot k-1}-\frac{1}{3\cdot k}\right)=$$

$$\sum_{k=1}^{n}\int_0^1(x^{3\cdot k-2}-x^{3\cdot k-1})\cdot dx$$

$$=\sum_{k=1}^{n}\int_0^1 x^{3\cdot k-2}\cdot(1-x)\cdot dx=\int_0^1\sum_{k=1}^{n}x^{3\cdot k-2}\cdot(1-x)\cdot dx=$$

$$\int_0^1(1-x)\cdot\sum_{k=1}^{n}x^{3\cdot k-2}\cdot dx$$

$$=\int_0^1(1-x)\cdot x\cdot\frac{1-x^{3\cdot n}}{1-x^3}\cdot dx=\int_0^1\frac{x\cdot(1-x^{3\cdot n})}{1+x+x^2}\cdot dx=$$

$$\int_0^1\frac{x}{1+x+x^2}\cdot dx-\int_0^1\frac{x^{3\cdot n}}{1+x}\cdot dx. \qquad (2.22)$$

Passing to the limit, when $n\to +\infty$, in equalities (2.22), according to (Vălcan, (I), 2016, Proposition), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k-1)}=\int_0^1\frac{x}{1+x+x^2}\cdot dx=\frac{1}{2}\cdot\ln 3-\frac{\pi\cdot\sqrt{3}}{18}. \qquad (2.23)$$

On the other hand, from the equalities in (2.21), it follows that:

$$\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k-1)}=\sum_{k=1}^{n}\left(\frac{1}{3\cdot k-1}-\frac{1}{3\cdot k}\right)$$

$$= \sum_{k=1}^{n}\frac{1}{3\cdot k-1} - \frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{k} =$$

$$\left(\frac{1}{3\cdot 1-1}+\frac{1}{3\cdot 2-1}+...+\frac{1}{3\cdot n-1}\right) - \frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}\right)$$

$$= \frac{1}{3}\cdot\left(\frac{1}{1-\frac{1}{3}}+\frac{1}{2-\frac{1}{3}}+...+\frac{1}{n-\frac{1}{3}}\right) - \frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}\right)$$

$$= \frac{1}{3}\cdot\left(\frac{1}{1-\frac{1}{3}}+\frac{1}{2-\frac{1}{3}}+...+\frac{1}{n-\frac{1}{3}}-\ln\left(n-\frac{1}{3}\right)\right) - \frac{1}{2}\cdot$$

$$\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}-\ln n\right)$$

$$+\frac{1}{3}\cdot\ln\left(n-\frac{1}{3}\right) - \frac{1}{3}\cdot\ln n = \frac{1}{3}\cdot(c_{n,-\frac{1}{3}} - c_n) + \frac{1}{3}\cdot\ln\frac{3\cdot n-1}{3\cdot n}. \tag{2.24}$$

Passing to the limit, when $n \to +\infty$, in equalities (2.24), we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k-1)} = \frac{1}{3}\cdot(c_{-\frac{1}{3}} - c_0). \tag{2.25}$$

Now, from equalities (2.20) and (2.25), we obtain the value of $c_{-\frac{1}{3}}$ from (1.61).

Now, knowing the value of $c_{-\frac{1}{3}}$, from the equalities in (1.61), we can calculate the limit of a string, otherwise. Thus, we have the equalities:

$$\sum_{k=1}^{n}\frac{1}{k\cdot(3\cdot k-1)} = \sum_{k=1}^{n}\left(\frac{3}{3\cdot k-1}-\frac{1}{k}\right)$$

$$= 3\cdot\sum_{k=1}^{n}\frac{1}{3\cdot k-1} - \sum_{k=1}^{n}\frac{1}{k} = 3\cdot$$

$$\left(\frac{1}{3\cdot 1-1}+\frac{1}{3\cdot 2-1}+...+\frac{1}{3\cdot n-1}\right) - \left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}\right)$$

$$= 3\cdot\frac{1}{3}\cdot\left(\frac{1}{1-\frac{1}{3}}+\frac{1}{2-\frac{1}{3}}+...+\frac{1}{n-\frac{1}{3}}\right) - c_n + \ln n$$

$$= c_{n,-\frac{1}{3}} + \ln\frac{3}{3\cdot n-1} - c_n + \ln n = c_{n,-\frac{1}{3}} - c_n + \ln\frac{3\cdot n}{3\cdot n-1}. \tag{2.26}$$

Passing to the limit, when $n \to +\infty$, in equalities (2.26), according to the second equality from (1.61), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(3\cdot k-1)} = \frac{3}{2}\cdot\ln 3 - \frac{\pi\cdot\sqrt{3}}{6}. \tag{2.20′}$$

5) $\alpha = \frac{2}{3}$. In (Vălcan, 2024) we proved that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+2)} = \frac{1}{4}\cdot\left(1-\ln 3+\frac{\pi\cdot\sqrt{3}}{9}\right). \tag{2.27}$$

But, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(3\cdot k)\cdot(3\cdot k+2)} = \frac{1}{2}\cdot\frac{1}{3\cdot k} - \frac{1}{2}\cdot\frac{1}{3\cdot k+1}.$$

So,

$$\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+2)} = \sum_{k=1}^{n}\left(\frac{1}{3\cdot k}-\frac{1}{3\cdot k+2}\right)$$

$$= \frac{1}{6}\cdot\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{2}\cdot\sum_{k=1}^{n}\frac{1}{3\cdot k+2} = \frac{1}{2}\cdot\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}\right) - \frac{1}{2}\cdot$$

$$\left(\frac{1}{3\cdot 1+2}+\frac{1}{3\cdot 2+2}+...+\frac{1}{3\cdot n+2}\right)$$

$$= \frac{1}{6}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \frac{1}{6}\cdot$$

$$\left(\frac{1}{1+\frac{2}{3}}+\frac{1}{2+\frac{2}{3}}+\cdots+\frac{1}{n+\frac{2}{3}}\right)$$

$$= \frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+...+\frac{1}{n}-\ln n\right) - \frac{1}{6}\cdot$$

$$\left(\frac{1}{1+\frac{2}{3}}+\frac{1}{2+\frac{2}{3}}+...+\frac{1}{n+\frac{2}{3}}-\ln\left(n+\frac{3}{3}\right)\right)$$

$$+\frac{1}{6}\cdot\ln n-\frac{1}{6}\cdot\ln\left(n+\frac{2}{3}\right)=\frac{1}{6}\cdot(c_n+\ln n)-\frac{1}{6}\cdot\left(c_{n,\frac{2}{3}}+\ln\left(n+\frac{2}{3}\right)\right)=\frac{1}{6}\cdot(c_n-c^{n,\frac{2}{3}})+\frac{1}{6}\cdot\ln\left(\frac{3\cdot n}{3\cdot n+2}\right). \qquad (2.28)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.28), we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+2)}=\frac{1}{6}\cdot(c_n-c^{n,\frac{2}{3}}). \qquad (2.29)$$

Now, from equalities (2.16) and (2.18), we obtain the value of $c^{\frac{2}{3}}$ from (1.65).

Moreover, still starting from equality (2.27), we obtain, otherwise, the value of $c^{-\frac{1}{3}}$. Thus, we have the equalities:

$$\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k+2)}=\frac{1}{2}\cdot\sum_{k=1}^{n}\left(\frac{1}{3\cdot k}-\frac{1}{3\cdot k+2}\right)=\frac{1}{2}\cdot\left(\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{3\cdot n}\right)-\frac{1}{2}\cdot\left(\frac{1}{5}+\frac{1}{8}+\cdots+\frac{1}{3\cdot n+2}\right)=$$

$$=\frac{1}{6}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\frac{1}{6}\cdot\left(\frac{1}{1-\frac{1}{3}}+\frac{1}{2-\frac{1}{3}}+\cdots+\frac{1}{n+1-\frac{1}{3}}-\frac{1}{1-\frac{1}{3}}\right)=$$

$$=\frac{1}{4}+\frac{1}{6}\cdot(c_n+\ln n)-\frac{1}{6}\cdot\left(c_{n+1,-\frac{1}{3}}+\ln\left(n+1-\frac{1}{3}\right)\right)=\frac{1}{4}+\frac{1}{6}\cdot(c_n-c^{n+1,-\frac{1}{3}})+\frac{1}{6}\cdot\ln\left(\frac{3\cdot n}{3\cdot n+2}\right). \qquad (2.30)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.30), we obtain:

$$\frac{1}{4}\cdot\left(1-\ln 3+\frac{\pi\cdot\sqrt{3}}{9}\right)=\frac{1}{4}+\frac{1}{6}\cdot(c-c^{-\frac{1}{3}});$$

whence we obtain that:

$$c^{-\frac{1}{3}}=c+\frac{3}{2}\cdot\ln 3-\frac{\pi\cdot\sqrt{3}}{6}.$$

Now, knowing the value of $c^{\frac{2}{3}}$, from the equalities in (1.65), we can calculate the limit of a string, otherwise. Thus, we have the equalities:

$$\sum_{k=1}^{n}\frac{1}{k\cdot(3\cdot k+2)}=\frac{1}{2}\cdot\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{3}{3\cdot k+2}\right)=\frac{1}{2}\cdot\sum_{k=1}^{n}\frac{1}{k}-\frac{3}{2}\cdot\sum_{k=1}^{n}\frac{1}{3\cdot k+2}=\frac{1}{2}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\frac{3}{2}\cdot\left(\frac{1}{3\cdot 1+2}+\frac{1}{3\cdot 2+2}+\ldots+\frac{1}{3\cdot n+2}\right)$$

$$=\frac{1}{2}\cdot c_n+\frac{1}{2}\cdot\ln n-\frac{3}{2}\cdot\frac{1}{3}\cdot\left(\frac{1}{1+\frac{2}{3}}+\frac{1}{2+\frac{2}{3}}+\ldots+\frac{1}{n+\frac{2}{3}}\right)$$

$$=\frac{1}{2}\cdot c_n+\frac{1}{2}\cdot\ln n-\frac{1}{2}\cdot c_{n,\frac{2}{3}}-\frac{1}{2}\cdot\ln(3\cdot n+2)=\frac{1}{2}\cdot(c_n-c_{n,\frac{2}{3}})+\ln\frac{3\cdot n}{3\cdot n+2}. \qquad (2.31)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.31), according to the first equality from (1.65), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(3\cdot k+2)}=\frac{3}{4}\cdot\left(1-\ln 3+\frac{\pi\cdot\sqrt{3}}{9}\right). \qquad (2.27')$$

6) $\alpha=-\frac{2}{3}$. First let us prove that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k-2)}=\frac{1}{4}\cdot\ln 3+\frac{\pi\cdot\sqrt{3}}{36}. \qquad (2.32)$$

For every $k\in\mathbf{N}^*$,

$$\frac{1}{(3\cdot k)\cdot(3\cdot k-2)}=\frac{1}{2}\cdot\frac{1}{3\cdot k-2}-\frac{1}{2}\cdot\frac{1}{3\cdot k}=\frac{1}{2}\cdot\int_0^1(x^{3\cdot k-3}-x^{3\cdot k-1})\cdot dx. \qquad (2.33)$$

So,

$$\sum_{k=1}^{n}\frac{1}{(3\cdot k)\cdot(3\cdot k-2)}=\frac{1}{2}\cdot\sum_{k=1}^{n}\left(\frac{1}{3\cdot k-2}-\frac{1}{3\cdot k}\right)=\frac{1}{2}\cdot\sum_{k=1}^{n}\int_0^1(x^{3\cdot k-3}-x^{3\cdot k-1})\cdot dx$$

$$= \frac{1}{2} \cdot \sum_{k=1}^{n} \int_0^1 x^{3\cdot k-3} \cdot (1-x^2) \cdot dx = \frac{1}{2} \cdot \int_0^1 \sum_{k=1}^{n} x^{3\cdot k-3} \cdot (1-x^2) \cdot dx = \frac{1}{2} \cdot \int_0^1 (1-x^2) \cdot \sum_{k=1}^{n} x^{3\cdot k-3} \cdot dx$$

$$= \frac{1}{2} \cdot \int_0^1 (1-x) \cdot (1+x) \cdot \frac{1-x^{3\cdot n}}{1-x^3} \cdot dx = \frac{1}{2} \cdot \int_0^1 \frac{(1+x)\cdot(1-x^{3\cdot n})}{1+x+x^2} \cdot dx$$

$$= \frac{1}{2} \cdot \int_0^1 \frac{1+x}{1+x+x^2} \cdot dx - \frac{1}{2} \cdot \int_0^1 \frac{x^{3\cdot n}}{1+x+x^2} \cdot dx. \quad (2.34)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.34), according to (Vălcan, (I), 2016, Proposition), we obtain that:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(3\cdot k)\cdot(3\cdot k-2)} = \frac{1}{2} \cdot \int_0^1 \frac{1+x}{1+x+x^2} \cdot dx = \frac{1}{4} \cdot \ln 3 + \frac{\pi\cdot\sqrt{3}}{36}. \quad (2.35)$$

On the other hand, from the equalities in (2.33), it follows that:

$$\sum_{k=1}^{n} \frac{1}{(3\cdot k)\cdot(3\cdot k-2)} = \frac{1}{2} \cdot \sum_{k=1}^{n} \left(\frac{1}{3\cdot k-2} - \frac{1}{3\cdot k}\right) = \frac{1}{2} \cdot \sum_{k=1}^{n} \frac{1}{3\cdot k-2} - \frac{1}{6} \cdot \sum_{k=1}^{n} \frac{1}{k} =$$

$$= \frac{1}{2} \cdot \left(\frac{1}{3\cdot 1-2} + \frac{1}{3\cdot 2-2} + \ldots + \frac{1}{3\cdot n-2}\right) - \frac{1}{6} \cdot \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right)$$

$$= \frac{1}{6} \cdot \left(\frac{1}{1-\frac{2}{3}} + \frac{1}{2-\frac{2}{3}} + \ldots + \frac{1}{n-\frac{2}{3}}\right) - \frac{1}{6} \cdot \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right)$$

$$= \frac{1}{6} \cdot \left(\frac{1}{1-\frac{2}{3}} + \frac{1}{2-\frac{2}{3}} + \ldots + \frac{1}{n-\frac{2}{3}} - \ln\left(n-\frac{2}{3}\right)\right) - \frac{1}{6} \cdot \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right)$$

$$+ \frac{1}{6} \cdot \ln\left(n-\frac{2}{3}\right) - \frac{1}{6} \cdot \ln n = \frac{1}{6} \cdot (c_{n,-\frac{2}{3}} - c_n) + \frac{1}{6} \cdot \ln \frac{3\cdot n-2}{3\cdot n}. \quad (2.36)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.36), we obtain:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(3\cdot k)\cdot(3\cdot k-2)} = \frac{1}{6} \cdot (c_{-\frac{2}{3}} - c_0). \quad (2.37)$$

Now, from equalities (2.32) and (2.37), we obtain the value of $c_{-\frac{2}{3}}$ from (1.65).

Now, knowing the value of $c_{-\frac{2}{3}}$, from the equalities in (1.65), we can calculate the limit of a string, otherwise. Thus, we have the equalities:

$$\sum_{k=1}^{n} \frac{1}{k\cdot(3\cdot k-2)} = \frac{1}{2} \cdot \sum_{k=1}^{n} \left(\frac{3}{3\cdot k-2} - \frac{1}{k}\right) = \frac{3}{2} \cdot \sum_{k=1}^{n} \frac{1}{3\cdot k-2} - \frac{1}{2} \cdot \sum_{k=1}^{n} \frac{1}{k} =$$

$$= \frac{3}{2} \cdot \left(\frac{1}{3\cdot 1-2} + \frac{1}{3\cdot 2-2} + \ldots + \frac{1}{3\cdot n-2}\right) - \frac{1}{2} \cdot \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{1-\frac{2}{3}} + \frac{1}{2-\frac{2}{3}} + \ldots + \frac{1}{n-\frac{2}{3}}\right) - \frac{1}{2} \cdot c_n + \frac{1}{2} \cdot \ln n$$

$$= \frac{1}{2} \cdot c_{n,-\frac{2}{3}} + \frac{1}{2} \cdot \ln \frac{3}{3\cdot n-2} - \frac{1}{2} \cdot c_n + \frac{1}{2} \cdot \ln n = \frac{1}{2} \cdot (c_{n,-\frac{2}{3}} - c_n) + \frac{1}{2} \cdot \ln \frac{3\cdot n}{3\cdot n-2}. \quad (2.38)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.38), according to the second equality from (1.65), we obtain that:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k\cdot(3\cdot k-2)} = \frac{3}{4} \cdot \ln 3 + \frac{\pi\cdot\sqrt{3}}{12}. \quad (2.32')$$

More than that,

$$c_{n,\frac{1}{3}}=\frac{1}{1+\frac{1}{3}}+\frac{1}{2+\frac{1}{3}}+\ldots+\frac{1}{n+\frac{1}{3}}-\ln\left(n+\frac{1}{3}\right)$$
$$=\frac{1}{1-\frac{2}{3}}+\frac{1}{2-\frac{2}{3}}+\ldots+\frac{1}{n-\frac{2}{3}}-\ln\left(n-\frac{2}{3}\right)+\frac{1}{n+\frac{1}{3}}-\ln\left(n+\frac{1}{3}\right)+\ln\left(n-\frac{2}{3}\right)-3=c_{n,-\frac{2}{3}}+\frac{1}{n+\frac{1}{3}}+\ln\left(\frac{3\cdot n-2}{3\cdot n+1}\right)-3. \quad (2.39)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.39), we obtain:

$$c^{\frac{1}{3}}=c^{-\frac{2}{3}}-3, \quad (2.40)$$

equality proven by equalities (1.61) and (1.65).

And,

$$c_{n,-\frac{1}{3}}=\frac{1}{1-\frac{1}{3}}+\frac{1}{2-\frac{1}{3}}+\ldots+\frac{1}{n-\frac{1}{3}}-\ln\left(n-\frac{1}{3}\right)$$
$$=\frac{1}{1+\frac{2}{3}}+\frac{1}{2+\frac{2}{3}}+\ldots+\frac{1}{n+\frac{2}{3}}-\ln\left(n+\frac{2}{3}\right)-\frac{1}{n-\frac{1}{3}}-\ln\left(n-\frac{1}{3}\right)+\ln\left(n+\frac{2}{3}\right)+\frac{3}{2}$$
$$=c_{n,\frac{2}{3}}-\frac{1}{n-\frac{1}{3}}+\ln\left(\frac{3\cdot n+2}{3\cdot n-1}\right)+\frac{3}{2}. \quad (2.41)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.41, we obtain:

$$c^{-\frac{1}{3}}=c^{\frac{2}{3}}+\frac{3}{2}, \quad (2.42)$$

equality proven by equalities (1.61) and (1.65).

The attentive reader interested in these issues will notice that the condition that the number α belongs to the interval (0,1) is imposed only so that all terms of the sum $\frac{1}{1-\alpha}+\frac{1}{2-\alpha}+\frac{1}{3-\alpha}+\ldots+\frac{1}{n-\alpha}$ are positive. Neither Proposition 1 nor Observations 2 impose such a condition. Therefore, next, we will see what happens if $\alpha\in[1,+\infty)$, keeping the notations above.

7) $\alpha=k\in\mathbf{N}^*$. We approach this case inductively. For example:

$$c_{n,1}=\frac{1}{1+1}+\frac{1}{2+1}+\frac{1}{3+1}+\ldots+\frac{1}{n+1}-\ln(n+1)$$
$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n+1}-\ln(n+1)-1=c_n-1. \quad (2.43)$$

$$c_{n,2}=\frac{1}{1+2}+\frac{1}{2+2}+\frac{1}{3+2}+\ldots+\frac{1}{n+2}-\ln(n+2)$$
$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n+2}-\ln(n+2)-1-\frac{1}{2}=c_n-\left(1+\frac{1}{2}\right). \quad (2.43')$$

$$c_{n,3}=\frac{1}{1+3}+\frac{1}{2+3}+\frac{1}{3+3}+\ldots+\frac{1}{n+3}-\ln(n+3)$$
$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n+3}-\ln(n+3)-1-\frac{1}{2}-\frac{1}{3}=c_n-\left(1+\frac{1}{2}+\frac{1}{3}\right). \quad (2.43'')$$

...

$$c_{n,k}=\frac{1}{1+k}+\frac{1}{2+k}+\frac{1}{3+k}+\ldots+\frac{1}{n+k}-\ln(n+k)$$
$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n+k}-\ln(n+k)-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{k}$$
$$=c_n-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}\right). \quad (2.43^{(k-1)})$$

We observe that, for a fixed number $k\in\mathbf{N}^*$, $c_{n,k}$ is a convergent sequence and:

$$\lim_{n\to\infty}c_{n,k}=\lim_{n\to\infty}\left(\frac{1}{1+k}+\frac{1}{2+k}+\frac{1}{3+k}+\cdots+\frac{1}{n+k}-\ln(n+k)\right)\overset{not.}{=}c_k$$
$$=c_0-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}\right)<0. \quad (2.44)$$

Of course we can determine c_α, for a number $\alpha\in\mathbf{Q}\setminus\mathbf{N}$, $\alpha>1$. Thus:

8) $\alpha=\frac{3}{2}$. In (Vălcan, 2024) we proved that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+3)}=\frac{1}{3}\cdot\left(\frac{4}{3}-\ln 2\right). \qquad (2.45)$$

But, for every $k\in\mathbf{N}^*$,

$$\frac{1}{(2\cdot k)\cdot(2\cdot k+3)}=\frac{1}{3}\cdot\left(\frac{1}{2\cdot k}-\frac{1}{2\cdot k+3}\right).$$

So,

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+3)}=\frac{1}{3}\cdot\sum_{k=1}^{n}\left(\frac{1}{2\cdot k}-\frac{1}{2\cdot k+3}\right)$$

$$=\frac{1}{6}\cdot\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k+3}=\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\frac{1}{3}\cdot\left(\frac{1}{2\cdot1+3}+\frac{1}{2\cdot2+3}+\ldots+\frac{1}{2\cdot n+3}\right)$$

$$=\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\frac{1}{6}\cdot\left(\frac{1}{1+\frac{3}{2}}+\frac{1}{2+\frac{3}{2}}+\ldots+\frac{1}{n+\frac{3}{2}}\right)$$

$$=\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)-\frac{1}{6}\cdot\left(\frac{1}{1+\frac{3}{2}}+\frac{1}{2+\frac{3}{2}}+\ldots+\frac{1}{n+\frac{3}{2}}-\ln\left(n+\frac{3}{2}\right)\right)$$

$$+\frac{1}{6}\cdot\ln n-\frac{1}{6}\cdot\ln\left(n+\frac{3}{2}\right)=\frac{1}{6}\cdot(c_n-c_{n,\frac{3}{2}})+\frac{1}{6}\cdot\ln\frac{2\cdot n}{2\cdot n+3}. \qquad (2.46)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.46), we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+3)}=\frac{1}{6}\cdot(c_0-c^{\frac{3}{2}}). \qquad (2.47)$$

Now, from equalities (2.45) and (2.47), we obtain that:

$$c^{\frac{3}{2}}=c_0-2\cdot\left(\frac{4}{3}-\ln 2\right). \qquad (2.48)$$

Otherwise: The following equalities hold:

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+3)}=\frac{1}{3}\cdot\sum_{k=1}^{n}\left(\frac{1}{2\cdot k}-\frac{1}{2\cdot k+3}\right)$$

$$=\frac{1}{6}\cdot\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k+3}=\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\frac{1}{3}\cdot\left(\frac{1}{2\cdot1+3}+\frac{1}{2\cdot2+3}+\ldots+\frac{1}{2\cdot n+3}\right)$$

$$=\frac{1}{6}\cdot c_n+\frac{1}{6}\cdot\ln n-\frac{1}{3}\cdot\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2\cdot n+3}\right)+\frac{1}{3}\cdot\left(1+\frac{1}{3}\right)=\frac{1}{6}\cdot c_n+\frac{1}{6}\cdot\ln n-\frac{1}{3}\cdot\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2\cdot n+2}+\frac{1}{2\cdot n+3}\right)+\frac{1}{3}\cdot\left(1+\frac{1}{3}\right)$$

$$+\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n+1}\right)=\frac{1}{6}\cdot c_n+\frac{1}{6}\cdot\ln n-\frac{1}{3}\cdot\left(1+\frac{1}{2}+\ldots+\frac{1}{2\cdot n+2}+\frac{1}{2\cdot n+3}-\ln(2\cdot n+3)\right)+\frac{4}{9}$$

$$+\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n+1}-\ln(n+1)\right)-\frac{1}{3}\cdot\ln(2\cdot n+3)+\frac{1}{6}\cdot\ln(n+1)$$

$$=\frac{1}{6}\cdot c_n+\frac{1}{6}\cdot\ln n-\frac{1}{3}\cdot c_{2\cdot n+3}+\frac{4}{9}+\frac{1}{6}\cdot c_{n+1}-\frac{1}{3}\cdot\ln(2\cdot n+3)+\frac{1}{6}\cdot\ln(n+1)$$

$$=\frac{1}{6}\cdot c_n-\frac{1}{3}\cdot c_{2\cdot n+3}+\frac{1}{6}\cdot c_{n+1}+\frac{4}{9}+\frac{1}{6}\cdot\ln\frac{n\cdot(n+1)}{(2\cdot n+3)^2}. \qquad (2.49)$$

Passing to the limit, when $n\to+\infty$, in equalities (2.49), we obtain:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k+3)}=\frac{4}{9}-\frac{1}{3}\cdot\ln 2, \qquad (2.50)$$

because the sequence $(c_n)_{n\geq1}$ is convergent and, thus:

$$\lim_{n\to\infty}(c_n-2\cdot c_{2\cdot n+3}+c_{n+1})=0.$$

From equalities (2.47) and (2.50), we again obtain the value of $c^{\frac{3}{2}}$ from (2.48).

Now, knowing the value of $c^{\frac{3}{2}}$, from equality (2.48), we can calculate the limit of a string, otherwise. Thus, we have the equalities:

$$\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k+3)}=\frac{1}{3}\cdot\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2}{2\cdot k+3}\right)$$

$$= \frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{k} - \frac{2}{3}\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k+3} = \frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right) - \frac{2}{3}\cdot\left(\frac{1}{2\cdot 1+3}+\frac{1}{2\cdot 2+3}+\ldots+\frac{1}{2\cdot n+3}\right)$$

$$= \frac{1}{3}\cdot c_n + \frac{1}{3}\cdot\ln n - \frac{1}{3}\cdot\left(\frac{1}{1+\frac{3}{2}}+\frac{1}{2+\frac{3}{2}}+\ldots+\frac{1}{n+\frac{3}{2}} - \ln\left(n+\frac{2}{3}\right)\right) - \frac{1}{3}\cdot\ln\left(n+\frac{3}{2}\right) = \frac{1}{3}\cdot c_n - \frac{1}{3}\cdot c_{n,\frac{3}{2}} + \frac{1}{3}\cdot\ln\frac{2\cdot n}{2\cdot n+3}. \quad (2.51)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.51), according to equality (2.48), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k+3)} = \frac{2}{3}\cdot\left(\frac{4}{3}-\ln 2\right). \quad (2.45')$$

On the other hand,

$$\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k+3)} = \frac{1}{3}\cdot\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{2}{2\cdot k+3}\right)$$

$$= \frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{k} - \frac{2}{3}\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k+3} = \frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right) - \frac{2}{3}\cdot\left(\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2\cdot n+3}\right) = \frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right) - \frac{2}{3}\cdot\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2\cdot n+3}\right) + \frac{2}{3}\cdot\left(\frac{1}{1}+\frac{1}{3}\right)$$

$$= \frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right) - \frac{2}{3}\cdot\left(\frac{1}{2\cdot 1-1}+\frac{1}{2\cdot 2-1}+\frac{1}{2\cdot 3-1}+\ldots+\frac{1}{2\cdot(n+2)-1}\right) + \frac{2}{3}\cdot\left(\frac{1}{1}+\frac{1}{3}\right) = \frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right) - \frac{1}{3}\cdot\left(\frac{1}{1-\frac{1}{2}}+\frac{1}{2-\frac{1}{2}}+\frac{1}{3-\frac{1}{2}}+\ldots+\frac{1}{(n+2)-\frac{1}{2}}\right) + \frac{2}{3}\cdot\left(\frac{1}{1}+\frac{1}{3}\right)$$

$$= \frac{1}{3}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right) - \frac{1}{3}\cdot\left(\frac{1}{1-\frac{1}{2}}+\frac{1}{2-\frac{1}{2}}+\frac{1}{3-\frac{1}{2}}+\ldots+\frac{1}{(n+2)-\frac{1}{2}} - \ln\left(n+2-\frac{1}{2}\right)\right) + \frac{8}{9} + \frac{1}{3}\cdot\ln n - \frac{1}{3}\cdot\ln\frac{2\cdot n+3}{2} = \frac{1}{3}\cdot\left(c_n - c_{n,-\frac{1}{2}}\right) + \frac{8}{9} + \frac{1}{3}\cdot\ln\frac{2\cdot n}{2\cdot n+3}. \quad (2.52)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.52), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k+3)} == \frac{1}{3}\cdot\left(c_0 - c_{-\frac{1}{2}}\right) + \frac{8}{9}. \quad (2.53)$$

From equalities (2.45) and (2.53), it follows that:

$$\frac{1}{3}\cdot\left(c_0 - c_{\frac{3}{2}}\right) = \frac{1}{3}\cdot\left(c_n - c_{n,-\frac{1}{2}}\right) + \frac{8}{9},$$ that is:

$$c_{\frac{3}{2}} = c_{-\frac{1}{2}} - \frac{8}{3}. \quad (2.54)$$

Equalities (2.57) and (2.48) confirm that the equalities from (2.54) hold.

From equalities (2.45′) and (2.53), the second equality from (2.57) follows.

Now, a closer look at what was presented above prompts us to construct strings of the form $c_{n,-k}$, for a fixed number $k \in \mathbf{N}^*$.

9) $\alpha = -k \in \mathbf{N}^*$. We approach this case inductively. For example:

$$c_{n,-1} = \frac{1}{2-1}+\frac{1}{3-1}+\frac{1}{4-1}+\ldots+\frac{1}{n-1} - \ln(n-1)$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1} - \ln(n-1) = c_{n-1} = c_n - \frac{1}{n} + \ln\frac{n}{n-1}. \quad (2.55)$$

$$c_{n,-2} = \frac{1}{3-2}+\frac{1}{4-2}+\frac{1}{5-2}+\ldots+\frac{1}{n-2} - \ln(n-2)$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-2} - \ln(n-2)$$

$$=c_{n-2}-\left(\frac{1}{n-1}+\frac{1}{n}\right)+\ln\frac{n}{n-2}.$$ (2.55′)

$$c_{n,-3}=\frac{1}{4-3}+\frac{1}{5-3}+\frac{1}{6-3}+\ldots+\frac{1}{n-3}-\ln(n-3)$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-3}-\ln(n+3)$$

$$=c_{n-3}-\left(\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right)+\ln\frac{n}{n-3}.$$ (2.55″)

...

$$c_{n,-k}=\frac{1}{(k+1)-k}+\frac{1}{(k+2)-k}+\frac{1}{(k+3)-k}+\ldots+\frac{1}{n-k}-\ln(n-k)$$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-k}-\ln(n-k)$$

$$=c_{n-k}-\left(\frac{1}{n-k+1}+\ldots+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right)+\ln\frac{n}{n-k}.$$ ($2.55^{(k-1)}$)

We observe that, for every $k\in\mathbf{N}^*$, $c_{n,-k}$ is a convergent sequence and:

$$\lim_{n\to\infty}c_{n,-k}=\lim_{n\to\infty}\left(\frac{1}{(k+1)-k}+\frac{1}{(k+2)-k}+\frac{1}{(k+3)-k}+\cdots+\frac{1}{n-k}-\ln(n-k)\right)\overset{not.}{=}c_{-k}=c_0.$$ (2.56)

Of course we can also determine $c_{-\alpha}$, for a number $\alpha\in\mathbf{Q}\setminus\mathbf{N}$, with $\alpha>1$. Thus:

10) $\alpha=-\frac{3}{2}$. First let us show that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-3)}=\frac{1}{3}\cdot(\ln 2-1).$$ (2.57)

For every $k\in\mathbf{N}^*$, $k\geq 2$:

$$\frac{1}{(2\cdot k)\cdot(2\cdot k-3)}=\frac{1}{3}\cdot\left(\frac{1}{2\cdot k-3}-\frac{1}{2\cdot k}\right)=\frac{1}{3}\cdot\int_0^1(x^{2\cdot k-4}-x^{2\cdot k-1})\cdot dx.$$ (2.58)

So,

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-3)}=-\frac{1}{2}+\frac{1}{3}\cdot\sum_{k=2}^{n}\left(\frac{1}{2\cdot k-3}-\frac{1}{2\cdot k}\right)=-\frac{1}{2}+\frac{1}{3}\cdot\sum_{k=2}^{n}\int_0^1(x^{2\cdot k-4}-x^{2\cdot k-1})\cdot dx$$

$$=-\frac{1}{2}+\frac{1}{3}\cdot\sum_{k=2}^{n}\int_0^1(x^{2\cdot k-4}-x^{2\cdot k-1})\cdot dx=-\frac{1}{2}+\frac{1}{3}\cdot\sum_{k=2}^{n}\int_0^1 x^{2\cdot k-3}\cdot(1-x^3)\cdot dx$$

$$=-\frac{1}{2}+\frac{1}{3}\cdot\int_0^1\sum_{k=2}^{n}x^{2\cdot k-4}\cdot(1-x^3)\cdot dx=-\frac{1}{2}+\frac{1}{3}\cdot\int_0^1(1-x^3)\cdot\sum_{k=2}^{n}x^{2\cdot k-4}\cdot dx$$

$$=-\frac{1}{2}+\frac{1}{3}\cdot\int_0^1(1-x^3)\cdot\frac{1-x^{2\cdot n-2}}{1-x^2}\cdot dx=-\frac{1}{2}+\frac{1}{3}\cdot\int_0^1(1+x+x^2)\cdot\frac{1-x^{2\cdot n-2}}{1+x}\cdot dx$$

$$=-\frac{1}{2}+\frac{1}{3}\cdot\int_0^1\frac{1+x+x^2}{1+x}\cdot dx-\int_0^1\frac{(1+x+x^2)\cdot x^{2\cdot n-2}}{1+x}\cdot dx.$$ (2.59)

Passing to the limit, when $n\to+\infty$, in equalities (2.59), according to (Vălcan, 2016, Proposition), we obtain the equality from (2.57).

On the other hand, from the equalities in (2.58), it follows that:

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-3)}=\frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k-3}-\frac{1}{6}\cdot\sum_{k=1}^{n}\frac{1}{k}$$

$$=\frac{1}{3}\cdot\left(\frac{1}{2\cdot 1-3}+\frac{1}{2\cdot 2-3}+\ldots+\frac{1}{2\cdot n-3}\right)-\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\frac{1}{6}\cdot\left(\frac{1}{1-\frac{3}{2}}+\frac{1}{2-\frac{3}{2}}+\ldots+\frac{1}{n-\frac{3}{2}}\right)-\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\frac{1}{6}\cdot\left(\frac{1}{1-\frac{3}{2}}+\frac{1}{2-\frac{3}{2}}+\ldots+\frac{1}{n-\frac{3}{2}}-\ln\left(n-\frac{3}{2}\right)\right)-\frac{1}{6}\cdot$$

$$\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)+\frac{1}{6}\cdot\ln\left(n-\frac{3}{2}\right)-\frac{1}{6}\cdot\ln n$$

$$=\frac{1}{6}\cdot\left(c_{n,-\frac{3}{2}}-c_n\right)+\frac{1}{6}\cdot\ln\frac{2\cdot n-3}{2\cdot n}. \qquad (2.60)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.60), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-3)}=\frac{1}{6}\cdot\left(c_{n,-\frac{3}{2}}-c_n\right). \qquad (2.61)$$

Now, from equalities (2.57) and (2.61), we obtain that:

$$c_{-\frac{3}{2}}=c_0+2\cdot(\ln 2-1). \qquad (2.62)$$

But,

$$\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-3)}=\frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k-3}-\frac{1}{6}\cdot\sum_{k=1}^{n}\frac{1}{k}$$

$$=\frac{1}{3}\cdot\left(\frac{1}{2\cdot 1-3}+\frac{1}{2\cdot 2-3}+\ldots+\frac{1}{2\cdot n-3}\right)-\frac{1}{6}\cdot$$

$$\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)=\frac{1}{6}\cdot\left(\frac{1}{1-\frac{3}{2}}+\frac{1}{2-\frac{3}{2}}+\ldots+\frac{1}{n-\frac{3}{2}}\right)-\frac{1}{6}\cdot$$

$$\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)=-\frac{1}{3}+\frac{1}{6}\cdot\left(\frac{1}{1-\frac{1}{2}}+\ldots+\frac{1}{(n-1)-\frac{1}{2}}\right)-$$

$$\frac{1}{6}\cdot\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)=-\frac{1}{3}+\frac{1}{6}\cdot$$

$$\left(\frac{1}{1-\frac{1}{2}}+\ldots+\frac{1}{(n-1)-\frac{1}{2}}+\frac{1}{n-\frac{1}{2}}-\ln\left(n-\frac{1}{2}\right)\right)-\frac{1}{6}\cdot$$

$$\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)+\frac{1}{6}\cdot\ln\left(n-\frac{1}{2}\right)-\frac{1}{6}\cdot\ln n$$

$$=-\frac{1}{3}+\frac{1}{6}\cdot\left(c_{n,-\frac{1}{2}}-c_n\right)+\frac{1}{6}\cdot\ln\frac{2\cdot n-1}{2\cdot n}. \qquad (2.63)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.63), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2\cdot k)\cdot(2\cdot k-3)}=-\frac{1}{3}+\frac{1}{6}\cdot\left(c_{-\frac{1}{2}}-c_0\right). \qquad (2.64)$$

Now, from equalities (1.57), (2.60) and (2.63), we obtain the equality from (2.62), and, from equalities (1.57), (2.64) and (2.62), we obtain that:

$$c_{-\frac{3}{2}}=c_{-\frac{1}{2}}-2, \qquad (2.65)$$

which results from the equalities in (2.63).

Finally, knowing the value of $c_{-\frac{3}{2}}$, from the equality in (2.62), we can calculate the limit of a string, otherwise:

$$\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k-3)}=\frac{1}{3}\cdot\sum_{k=1}^{n}\left(\frac{2}{2\cdot k-3}-\frac{1}{k}\right)$$

$$=\frac{2}{3}\cdot\sum_{k=1}^{n}\frac{1}{2\cdot k-3}-\frac{1}{3}\cdot\sum_{k=1}^{n}\frac{1}{k}=\frac{2}{3}\cdot$$

$$\left(\frac{1}{2\cdot 1-3}+\frac{1}{2\cdot 2-3}+\ldots+\frac{1}{2\cdot n-3}\right)-\frac{1}{3}\cdot$$

$$\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

$$=\frac{1}{3}\cdot\left(\frac{1}{1-\frac{3}{2}}+\frac{1}{2-\frac{3}{2}}+\ldots+\frac{1}{n-\frac{3}{2}}\right)-\frac{1}{3}\cdot c_n+\frac{1}{3}\cdot\ln n$$

$$=\frac{1}{3}\cdot c_{n,-\frac{3}{2}}-\frac{1}{3}\cdot\ln\frac{2}{2\cdot n-3}-\frac{1}{3}\cdot c_n+\frac{1}{3}\cdot\ln n$$

$$=\frac{1}{3}\cdot\left(c_{n,-\frac{3}{2}}-c_n\right)+\frac{1}{2}\cdot\ln\frac{2\cdot n}{2\cdot n-3}. \qquad (2.66)$$

Passing to the limit, when $n \to +\infty$, in equalities (2.66), according to the equality from (2.62), we obtain that:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k\cdot(2\cdot k-3)}=\frac{2}{3}\cdot(\ln 2-1). \qquad (2.57')$$

4. Conclusions

Therefore, we can say that the expressions of the constants c_α and $c_{-\alpha}$ from the paper (Vălcan, 2024) are true and can also be obtained by other methods.

At the end of this paper, we propose, to the reader who is attentive and interested in these matters, to solve the following exercise:

Exercise: *According to the above model,*

1) *prove that the equalities from (1.75) and (1.76) hold and, in addition*:

$$c_{\frac{4}{3}} = c_0 - \frac{15}{4} + \frac{3}{2}\cdot \ln 3 + \frac{\pi\cdot\sqrt{3}}{6} \qquad \text{and}$$

$$c_{-\frac{4}{3}} = c_0 - \frac{15}{4} + \frac{3}{2}\cdot \ln 3 + \frac{\pi\cdot\sqrt{3}}{6} \qquad (2.66)$$

and:

$$c_{\frac{4}{3}} = c_{\frac{1}{3}} - \frac{3}{4} = c_{-\frac{2}{3}} - \frac{15}{4}, \qquad \text{and}$$

$$c_{-\frac{4}{3}} = c_{-\frac{1}{3}} - 3 = c_{\frac{2}{3}} - \frac{3}{2}. \qquad (2.67)$$

2) *without using the equalities from (1.44) and (1.50), show that, for every p, q*$\in$***N*****, p<q, the equalities hold*:

$$c_{\frac{p}{q}} = c_0 - q\cdot\int_0^1\left[\frac{x^{q-1}\cdot(1-x^p)}{1-x^q}\right]\cdot dx \qquad \text{and}$$

$$c_{-\frac{p}{q}} = c_0 + q\cdot\int_0^1\left[\frac{x^{q-p-1}\cdot(1-x^p)}{1-x^q}\right]\cdot dx \qquad (2.68)$$

and, if:

$$s = \left[\frac{q}{p}\right],$$

then:

$$c_{\frac{q}{p}} = c_{\frac{r}{p}} - \left(\frac{1}{1+\frac{r}{p}} + \frac{1}{2+\frac{r}{p}} + \frac{1}{s+\frac{r}{p}}\right) \qquad \text{and}$$

$$c_{-\frac{q}{p}} = c_{\frac{r}{p}} - \left(\frac{1}{1-\frac{r}{p}} + \frac{1}{2-\frac{r}{p}} + \frac{1}{s-\frac{r}{p}}\right). \qquad (2.69)$$

We also specify that for the calculation of all integrals we used the methods from (Fihtenholț, 1964).

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