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Theoretical article

From Diophantian Equations to Matrix Equations (III) - Other Diophantian Quadratic Equations and Diophantian Equations of Higher Degree

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Abstract

In this paper, we propose to continue the steps started in the first two papers with the same generic title and symbolically denoted by (I) and (II), namely, the presentation of ways of achieving a systemic vision on a certain mathematical notional content, a vision that to motivate and mobilize the activity of those who teach in the classroom, thus facilitating both the teaching and the assimilation of notions, concepts, scientific theories approached by the educational disciplines that present phenomena and processes from nature. Thus, we will continue in the same systemic approach, solving some Diophantine equations of higher degree, more precisely some generalizations of the Pythagorean equation and some quadratic Diophantine equations, in the set of natural numbers, then of the whole numbers, in order to "submerge" a such an equation in a ring of matrices and try to find as many matrix solutions as possible. In this way we will solve 12 large classes of Diophantine quadratic or higher order equations. For attentive readers interested in these matters, at the end of the paper we will propose 6 open problems. The solution of each of these open problems represents, in fact, a vast research activity and that can open the way to solving such more complicated Diophantine and / or matrix equations.

Keywords:
equation; system; solution; matrix; ring.

Zusammenfasung

In dieser Arbeit schlagen wir vor, das fortzusetzen, was wir in den ersten beiden Arbeiten mit demselben allgemeinen Titel und symbolisch mit (I) und (II) bezeichnet haben, nämlich die Lösung der diophantischen Gleichungen, indem wir sie in den Ring der Matrizen eintauchen $(\mathscr{M}_n(\mathbf{Z}),+,\cdot)$ und das Studium der Lösung der entsprechenden Matrixgleichungen. Auf diese Weise werden wir hier 12 große Klassen quadratischer oder höherer diophantischer Gleichungen lösen, nachdem wir uns in den ersten beiden Werken mit pythagoreischen Gleichungen und bestimmten Verallgemeinerungen davon befasst haben. Nach wie vor sind alle in dieser Arbeit vorgestellten diophantischen Gleichungen vollständig gelöst und die Lösbarkeit der entsprechenden Matrixgleichungen wird spezifiziert und veranschaulicht. Darüber hinaus werden wir für aufmerksame Leser, die sich für diese Themen interessieren, am Ende des Aufsatzes sechs offene Probleme vorschlagen. Die Lösung jedes dieser offenen Probleme ist in der Tat eine umfangreiche Forschungsaktivität und könnte den Weg für die Lösung solch komplizierterer diophantischer und/oder Matrixgleichungen ebnen. Die Idee besteht darin, den Leser für solche Recherchen zu begeistern und zu entwickeln.

Schlüsselworte:
Gleichung; System; Lösung; Matrix; Ring.

1. Introduction

As I stated above, in this paper we will continue the steps started in (Vălcan, 2019) and continued in (Vălcan, 2022), namely, the approach in a systemic view of the solution of some Diophantine equations and the study of the solvability of the corresponding matrix equations of immersing the Diophantine equations in the ring of matrices.

Thus, we will continue, in the same systemic approach, solving some Diophantine equations, more precisely some generalizations of the Pythagorean equation (other than those presented in the first two works) and some Diophantine equations of higher degree, in the set of natural numbers, then in the set of integers. Finally, we will submerge each such equation into a matrix ring and try to find as many matrix solutions as possible.

Therefore, this paper continues the ideas from the works (Vălcan, 2019) and (Vălcan, 2022). In this sense we will keep and continue not only the solution ideas and examples, but also the numbering of the results.

In this paper, we will first solve the negative Pythagorean equation (G) and then move on to some higher order equations: (H), (H_1), (H_2), (I), (J), (K), (L), (M), (N), (O), (P), (Q) and (R), which we will solve in the set of natural numbers and then of integer numbers, after which we will submerge them, on all in the ring $\mathscr{M}_n(\mathbf{Z})$.

And here, in solving Diophantine equations, we will use didactic methods, easily accessible to pupils and students, different from those presented in

(Andreescu & Andrica, 2002) or (Cucurezeanu, 2005), but based on ideas from there.

We also specify that we will use the knowledge related to the divisibility relation in the set of integers according to the model presented in (Acu, 2010).

2. Problem Statement

As is known from school practice, solving Diophantine equations often turns out to be quite difficult for pupils, students or teachers. The reasons are multiple: this topic does not appear explicitly in school curricula, but neither do teachers allocate special lessons to teach methods for solving these types of equations. On the other hand, due to the rather large number of types of Diophantine equations, of their solving methods, since secondary school, such equations appear, in particular, as applications of the divisibility relation - see (Vălcan, 2017).

Unfortunately, the lack of special courses aimed at solving Diophantine equations in Mathematics faculties also contributes to this unpleasant situation. So, it is possible that a graduate of such a faculty, who became a teacher, does not know how to solve certain types of Diophantine equations and thus will not be able to teach students how to solve such equations. It is useless to talk about the immersion of such equations in different rings. (Vălcan, 2022)

Therefore, these three papers come to reduce this shortcoming both for students and teachers.

3. Research Questions

In our research we will try to find answers to the following questions:

Can Pythagoras' equation be generalized in another way, as I generalized it in the paper (Vălcan, 2022)?

Do the new generalized Pythagorean equations have solutions in the set of integers? What about other higher order Diophantine equations?

If we immerse these generalized Pythagorean equations in the matrix ring, $\mathcal{M}_n(\mathbf{Z})$, do the new matrix equations have solutions in this ring?

4. Purpose of the Study

Therefore, we will generalize, in other ways than those presented in (Vălcan, 2022), the well-known Pythagorean equation:

$$x^2+y^2=z^2, \tag{A}$$

we will solve in the set **Z** all these generalized Pythagorean equations, but also other Diophantine equations of higher degree and we will study their solvability in a matrix ring.

5. Research Methods

We start with the "*negative*" Pythagorean equation:

$$x^{-2}+y^{-2}=z^{-2}. \tag{G}$$

Theorem 7.1: *Equation (G) is solvable both in the set of natural numbers and in the set of integers*.

Proof: The equation is equivalent to:

$$x^2+y^2=\left(\frac{x\cdot y}{z}\right)^2. \tag{7.1}$$

This shows that if (x,y,z) is a solution of equation (7.1), then $z \mid (x\cdot y)$ and x^2+y^2 is a perfect square. Then equation (7.1) becomes:

$$x^2+y^2=t^2, \tag{7.2}$$

with $t\in\mathbf{N}^*$, and from the equalities (7.1) and (7.2), we obtain the equality:

$$t=\frac{x\cdot y}{z}. \tag{7.3}$$

We consider,

$$d=(x,y,t). \tag{7.4}$$

Then:

$$x=a\cdot d,\ y=b\cdot d \text{ and } t=c\cdot d, \tag{7.5}$$

where $a, b, c\in\mathbf{N}^*$, with:

$$(a,b,c)=1. \tag{7.6}$$

From the equalities (7.5), it follows that the equation (7.3) reduces to:

$$z=\frac{a\cdot b\cdot d}{c}. \tag{7.7}$$

Now, from the equalities (7.5) and (7.2), we obtain that:

$$a^2+b^2=c^2, \tag{7.8}$$

that is, the numbers a, b, c are two by two prime to each other – see the equality (7.6). So, from the equality (7.7), it is obtained that $c \mid d$, that is:

$d=k\cdot c,$

with $k\in\mathbf{N}^*$. Therefore,

$$x=k\cdot a\cdot c,\ y=k\cdot b\cdot c,\ t=k\cdot c^2,\ \text{and } z=k\cdot a\cdot b. \tag{7.9}$$

Now, from the equality (7.8) and Theorem 2.5 from (Vălcan, 2019), we obtain that:

$$a=m^2-n^2,\ b=2\cdot m\cdot n\ \text{and } c=m^2+n^2, \tag{7.10}$$

where m, $n\in\mathbf{N}^*$ and m>n. Finally, from the equalities (7.9) and (7.10), we obtain that the integer solutions of the equation (G) are given by the equalities:

$$x=k\cdot(m^4-n^4),\ y=2\cdot k\cdot m\cdot n(m^2+n^2)\ \text{and}$$
$$z=2\cdot k\cdot m\cdot n(m^2-n^2), \tag{7.11}$$

where k, m, $n\in\mathbf{Z}$ and m>n. □

Let us now consider equation (G) in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$:

$$X^{-2}+Y^{-2}=Z^{-2}. \tag{G′}$$

Theorem 7.2: *The equation (G′) is not solvable in the ring* $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$.

Proof: We assume that the equation (G′) is solvable in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$. Then, a solution $(X,Y,Z)\in\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})$ of the equation (G′) consists of invertible matrices. If we note:

$$X^{-1}=M,\ Y^{-1}=N\ \text{and } Z^{-1}=P,$$

then the matrix equation (G′) becomes:

$$M^2+N^2=P^2, \tag{A″}$$

which, according to Theorem 3.1 from (Vălcan, 2019), is solvable in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$, because the matrices M, N, $P\in\mathcal{M}_n(\mathbf{Z})$ can be solutions of the equation (A″), if:

$$M=A^2-B^2,\ N=2\cdot A\cdot B\ \text{and } P=A^2+B^2, \tag{3.3}$$

where A, $B\in\mathcal{M}_n(\mathbf{Z})$ and:

$$A\cdot B=B\cdot A. \tag{3.1′}$$

In this case:

$$X=(A^2-B^2)^{-1},\quad Y=(2\cdot A\cdot B)^{-1}\quad \text{and}$$
$$Z=(A^2+B^2)^{-1}. \tag{3.3}$$

But, since A, $B\in\mathcal{M}_n(\mathbf{Z})$, it is possible that X or Z no longer belongs to the set $\mathcal{M}_n(\mathbf{Z})$. More than that, because:

$$\det(2\cdot A\cdot B)=2^n\cdot\det(A\cdot B),$$
$$\text{and } (2\cdot A\cdot B)^*=2^{n-1}\cdot(A\cdot B)^*,$$

it follows that:

$$(2\cdot A\cdot B)^{-1}=\frac{1}{2}\cdot(A\cdot B)^{-1}\notin\mathcal{M}_n(\mathbf{Z}).$$

In conclusion, it follows that this equation is not solvable in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$. □

We now pass to another equation:

Theorem 7.3: *The equation*:

$$x^4+y^4=z^2. \tag{H}$$

it is not solvable in the set of non-zero integers.

Proof: It suffices to consider x, y, $z\in\mathbf{N}^*$. We assume, by absurdity, that the equation (H) is solvable in non-zero natural numbers and we consider (x_1,y_1,z_1) a solution, with minimal z_1. We can suppose that:

$$(x_1,y_1,z_1)=1 \tag{7.12}$$

and taking into account that (x_1^2,y_1^2,z_1) is a primitive Pythagorean triplet, it follows that:

$$(x_1,y_1)=(y_1,z_1)=(z_1,x_1)=1 \tag{7.13}$$

and that x_1 and y_1 have different parities. To make a choice, assume that x_1 is odd and y_1 is even. In this case, z_1 is odd. Then, it follows that:

$$(z_1-x_1^2,z_1+x_1^2)=2. \tag{7.14}$$

Indeed, if:

$$(z_1-x_1^2,z_1+x_1^2)=d,$$

then $d\mid(z_1-x_1^2)$ and $d\mid(z_1+x_1^2)$. It follows that $d\mid(2\cdot z_1)$ and $d\mid(2\cdot x_1^2)$. From the last equality from (7.13) and from the fact that z_1 is odd, we obtain that:

$$d=2. \tag{7.15}$$

Because:

$$y_1^4=(z_1-x_1^2)\cdot(z_1+x_1^2), \tag{7.16}$$

from equality (7.14), it follows that one of the numbers $z_1-x_1^2$ and $z_1+x_1^2$ is divisible by 2, but not divisible by 4, so the second should be divisible by 8. Therefore:

$$y_1=2\cdot a\cdot b \tag{7.17}$$

and one of the following situations is verified:

$z_1-x_1^2=2\cdot a^4$, $z_1+x_1^2=8\cdot b^4$ or else $z_1-x_1^2=8\cdot b^4$,

$$z_1+x_1^2=2\cdot a^4, \tag{7.18}$$

where, in each case, a is odd and:

$$(a,b)=1. \tag{7.19}$$

The first two equalities in (7.18) cannot hold, since they would imply:

$$x_1^2=-a^4+4\cdot b^4, \tag{7.20}$$

which leads to the contradiction:

$a^4=\mathcal{M}_4+3$, that is

$1\equiv-1 \pmod 4$.

Therefore, we have the second alternative, that is:

$$z_1=a^4+4\cdot b^4, \tag{7.21}$$

with $0<a<z_1$ and, according to equality (7.21) and the last equality from (7.18), we obtain:

$$4\cdot b^4=z_1-a^4=(a^2-x_1)\cdot(a^2+x_1). \tag{7.22}$$

From the equality (7.19) and the last two equalities from (7.18), it follows that:

$$(a,x_1)=1 \tag{7.23}$$

and, we immediately obtain that:

$$(a^2-x_1,a^2+x_1)=2. \tag{7.24}$$

Consequently, from the equalities (7.22), we obtain that:

$$a^2-x_1=2\cdot x_2^4 \text{ and } a^2+x_1=2\cdot y_2^4, \tag{7.24}$$

where:

$x_2\cdot y_2=b$.

With the notation:

$a=z_2$,

from the equalities (7.24), we obtain that:

$$x_2^4+y_2^4=z_2^2,$$

with $0<z_2<z_1$, which contradicts the minimality of z_1.

***Otherwise*:** We will prove the Theorem using the waterfall method (see, Cucurezeanu, 2005, p. 119). We can assume here also that the equalities (7.13) hold and x is odd and y is even. The equation being Pythagorean, from Theorem 2.5 from (Vălcan, 2019), it follows that there are a and $b\in\mathbf{Z}$, such that:

$$x^2=a^2-b^2,\ y^2=2\cdot a\cdot b \text{ and } z=a^2+b^2, \tag{7.25}$$

with $a>b\geq0$, a – odd and b – even, but with:

$$(a,b)=1. \tag{7.26}$$

Because, from the first equality from (7.25), it follows that:

$$x^2+b^2=a^2,$$

and from the same Theorem 2.5 from (Vălcan, 2019), we deduce that:

$$x=p^2-q^2,\ b=2\cdot p\cdot q \text{ and } a=p^2+q^2, \tag{7.27}$$

with $p, q\in\mathbf{Z}^*$. From the second equality in (7.25) and the last two equalities in (7.27), it follows that:

$$y^2=4\cdot p\cdot q\cdot(p^2+q^2), \tag{7.28}$$

and:

$$(p,q)=1. \tag{7.29}$$

Now, from the equality (7.28), it follows that:

$$p=x_1^2,\ q=y_1^2,\ p^2+q^2=z_1^2 \text{ and } (x_1,y_1)=1. \tag{7.30}$$

From the first two equalities from (7.30) and the last equality from (7.27), we obtain (for $a=z_1$) that:

$$x_1^4+y_1^4=z_1^2. \tag{7.31}$$

So, to a solution (x,y,z) of the equation, with non-zero components, there corresponds another solution (x_1,y_1,z_1) also with non-zero components and vice versa. Moreover, in the new solution, according to the third equality from (7.30) and the last equalities from (7.27) and (7.25), we have:

$$z_1<\sqrt[4]{z}. \tag{7.32}$$

This resolution process can be repeated an infinite number of times, until we obtain a decreasing string of natural numbers z, z_1, z_2, ... So there exists an $n\in\mathbf{N}^*$ such that:

$$z_n=1,$$

in which case $x_n\cdot y_n=0$, which contradicts the hypothesis. □

Since the triplets of integer numbers, of the form $(0,k,k^2)$ or $(k,0,k^2)$ are solutions of equation (H), it follows that we have the following remark:

Remark 7.3: *Geometric interpretation: on the surface of the equation*:

$$z^2=x^4+y^4,$$

there are no integer coordinate points other than those located in the planes:

x=0 or y=0.

Regarding the matrix equation:

$$X^4+Y^4=Z^2, \tag{H′}$$

we have the following result:

Theorem 7.3: *The equation (H′) has no solutions (with non-zero components) in the ring* $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$.

Proof: Indeed, if this equation were solvable, then there would be matrices X, Y, $Z\in\mathcal{M}_n(\mathbf{Z})$ such that:

$$X^4+Y^4=Z^2.$$

Therefore, (X^2,Y^2,Z) is a triplet of Pythagorean matrices. Then, according to Theorem 3.1 from (Vălcan, 2019), there are matrices A, $B\in\mathcal{M}_n(Z)$, such that:

$$X^2=A^2-B^2,\ Y^2=2\cdot A\cdot B \text{ and } Z=A^2+B^2;$$

but, the second equality is impossible. Of course, the triplets of matrices with integer coefficients, of the form $(0,Y,Y^2)$ or $(X,0,X^2)$ are solutions of the equation (H′). □

From Theorem 7.2 it follows immediately:

Corollary 7.4: *The equation*:

$$x^4+y^4=z^4, \tag{I}$$

is not solvable in the set of nonzero integers; therefore, the only solutions of equation (I) are of the form (k,0,k) and (0,k,k), respectively, with $k\in\mathbf{Z}$. □

From Theorem 7.3, it follows:

Corollary 7.5: *The matrix equation*:

$$X^4+Y^4=Z^4, \tag{I′}$$

is not solvable in the ring of matrices $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$ *so, the only solutions of the equation (I′) are of the form (X,0,X) and (0,Y,Y), respectively, with X ,* $Y\in\mathcal{M}_n(\mathbf{Z})$. □

Corollary 7.5: *The only solutions of the equation*:

$$x^4+6\cdot x^2\cdot y^2+y^4=2\cdot z^2, \tag{J}$$

are given by the equalities:

$$x^2=y^2=k^2.$$

Proof: Indeed, if we replace in equation (J), x by u+v and y by u-v, we obtain the equation:

$$4\cdot u^4+4\cdot v^4=z^2, \tag{7.33}$$

whence it follows that z is even. Let be:

$$z=2\cdot t. \tag{7.34}$$

Then equation (J) becomes:

$$u^4+v^4=t^2, \tag{7.35}$$

what is an equation of the type (H). From Remark 7.3 it follows that the only solutions of equation (7.35) are obtained for:

u=0 or v=0.

So,

$$(u,v,t)\in\{(0,k,k^2),(k,0,k^2)\mid k\in\mathbf{Z}\}. \ \square$$

Following a proof path like that of the previous corollary, we obtain:

Corollary 7.5: *The only solutions of the matrix equation*:

$X^4+6\cdot X^2\cdot Y^2+Y^4=2\cdot Z^2$, (J′)

in the ring of matrices $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$, *are of the form* $(X,X,2\cdot X^2)$, *with* $X\in\mathcal{M}_n(\mathbf{Z})$. □

Two remarks are required here:

***Remarks 7.6*: 1)** *Equation (I) is a special case of Fermat's equation*:

$x^n+y^n=z^n$, (H₁)

where n>2 is a natural number and x, y, z are natural numbers.

2) *Euler conjectured that the equation*:

$x^n+y^n+z^n=t^n$ (H_2)

has no nonzero positive integer solutions, if n≥4. Noam Elkies (1988) gave the following counterexample:

$2682440^4+15365639^4+18796760^4=20615673^4$.

At the same time, Roger Frye (1988) found the smallest solution for the above equation,

$95800^4+217519^4+414560^4=422481^4$.

Theorem 7.7: *The matrix equation*:

$X^4+Y^4+Z^4=T^4$ (H_2')

has solutions in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$.

Proof: Of course, by Corollary 7.5, for any matrices, X, Y, Z, $T\in(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$, none of the quadruples (X,Y,0,T), (X,0,Z,T), respectively (0,Y,Z,T), are not solutions of the equation (H_2'). On the other hand, any quadruple of the form (X,0,0,X), or (0,X,0,X), or (0,0,X,X), is a solution of the equation (H_2'), if $X\in(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$. Now, we will look for other solutions of the equation (H_2'), different from the ones above. Let (x,y,z,t) be a quadruple of integers, solution of the equation (H_2) and consider the matrices:

$$X=\begin{pmatrix} x & 0\\ x & x\end{pmatrix},\qquad Y=\begin{pmatrix} y & 0\\ y & y\end{pmatrix},$$

$$Z=\begin{pmatrix} z & 0\\ z & z\end{pmatrix},\qquad \text{and}\qquad T=\begin{pmatrix} t & 0\\ t & t\end{pmatrix}.$$

Then,

$$X^4=\begin{pmatrix} x^4 & 0\\ 4\cdot x^4 & x^4\end{pmatrix},\qquad Y^4=\begin{pmatrix} y^4 & 0\\ 4\cdot y^4 & y^4\end{pmatrix},$$

$$Z^4=\begin{pmatrix} z^4 & 0\\ 4\cdot z^4 & z^4\end{pmatrix},\qquad \text{and}\qquad T^4=\begin{pmatrix} t^4 & 0\\ 4\cdot t^4 & t^4\end{pmatrix}.$$

So, according to the hypothesis,

$$X^4+Y^4+Z^4=\begin{pmatrix} x^4+y^4+z^4 & 0\\ 4\cdot(x^4+y^4+z^4) & x^4+y^4+z^4\end{pmatrix}$$

$$=\begin{pmatrix} t^4 & 0\\ 4\cdot t^4 & t^4\end{pmatrix}.$$

Next, we present the following result:

Theorem 7.7: *The equation*:

$x^4-y^4=z^2$, (K)

is not solvable in nonzero integers.

Proof: We can assume that x, y, $z\in\mathbf{N}^*$ and consider a solution (x,y,z), with:

$(x,y)=1$ (7.36)

and x minimal. Then the equation can still be written:

$x^4=y^4+z^2$, (K₁)

and (y^2,z,x^2) becomes a primitive Pythagorean triplet, so we have the following cases:

<u>Case 1:</u> According to Theorem 2.5 from (Vălcan, 2019), we have:

$y^2=a^2-b^2$, $z=2\cdot a\cdot b$ and $x^2=a^2+b^2$, (7.37)

where a>b>0 and:

$(a,b)=1$. (7.38)

From the equalities (7.37), it follows that:

$a^4-b^4=(x\cdot y)^2$

(7.39)

and a<x, in contradiction with the minimality of x.

Case 2: According to the same Theorem 2.5 from (Vălcan, 2019), we have:

$y^2=2\cdot a\cdot b$ $z=a^2-b^2$

and $x^2=a^2+b^2$, (7.40)

where a>b and:

(a.b)=1. (7.41)

Since (a,b,x) is also a primitive Pythagorean triplet (according to the last equality in (7.40)), we can assume that a is even and b is odd. Then, from the first equality in (7.40) and the equality (7.41), it follows that:

$a=2\cdot p^2$ and $b=q^2$. (7.42)

Then, from the last equality in (7.40), it follows that:

$x^2=4\cdot p^4+q^4$ and $y=2\cdot p\cdot q$. (7.43)

Therefore $(2\cdot p^2,q^2,x)$ is also a primitive triplet, so also Theorem 2.5 from (Vălcan, 2019) implies:

$p^2=r\cdot s$, $q^2=r^2-s^2$ and $x=r^2+s^2$, (7.44)

where r, s∈**N***, with the property that r>s and:

(r,s)=1. (7.45)

Finally, from the first equality from (7.44) we obtain that:

$r=u^2$ and $s=v^2$, (7.46)

where u, v∈N*, with:

(u,v)=1. (7.47)

So, from the equalities (7.46) and the second equality from (7.44), it follows that:

$u^4-v^4=q^2$ (K_2)

and, from the first equality in (7.46), the first equality in (7.44), the first equality in (7.42) and the first equality in (7.40), we obtain that:

$u=\sqrt{r}\leq p<2\cdot p^2<x$,

which contradicts the minimality of x. □

The following remarks are required here:

Remark 7.8: *Equation (K) has solutions for*:

$x^2=z=k^2$ and $y=0$.

or for:

$x=y=k$ and $z=0$. □ (7.48)

Reamrk 7.9: *For any matrix $X\in\mathcal{M}_n(\mathbf{Z})$, the triplets of the form*:

$(X,0,X^2)$ and $(X,X,0)$,

are solutions in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$, of the matrix equation:

$X^4-Y^4=Z^2$. (K′)

In the following, we will try to answer the question:

There exists a triplet of matrices $(X,Y,Z)\in\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})$, with $(X\neq0)\wedge(Y\neq0)\wedge(Z\neq0)$, such that solution of the equation (K′)?

In this sense we have the following result:

Theorem 7.10: *If the equation (B′) from (Vălcan, 2022) admits a solution (C,D,F,E), with*:

$C\cdot D=E\cdot F$ and $C^2+D^2+F^2=E^2$. (7.49)

then the triplet $(X,Y,Z)\in\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})$, with:

$X=E^2+F^2$, $Y=C^2-D^2$

and $Z=4\cdot C\cdot D\cdot(C^2+D^2)=4\cdot E\cdot F\cdot(E^2-F^2)$,

(7.50)

is the solution of the equation (K′).

Proof: If $(X,Y,Z)\in\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})$, is a solution of the equation (K′), then (Y^2,Z,X^2) is a Pythagorean triplet and, according to Theorem 3.1 from (Vălcan, 2019),

$Y^2=A^2-B^2$, $Z=2\cdot A\cdot B$ and $X^2=A^2+B^2$, (7.51)

with A, $B\in\mathcal{M}_n(\mathbf{Z})$. We deduce from this that: (Y,B,A) and (A,B,X) are Pythagorean triplets. So, according to the same Theorem 3.1 from (Vălcan, 2019), we have:

$$Y=C^2-D^2,\ B=2\cdot C\cdot D \text{ and } A=C^2+D^2, \tag{7.52}$$

and:

$$A=E^2-F^2,\ B=2\cdot E\cdot F \text{ and } X=E^2+F^2, \tag{7.53}$$

with C, D, E, $F\in\mathcal{M}_n(\mathbf{Z})$. From the equalities (7.52) and (7.53), it follows that the equalities (7.49) hold. On the other hand, from the equalities (7.51) and (7.52), it follows that:

$$X^4=C^8+38\cdot C^4\cdot D^4+12\cdot C^6\cdot D^2+12\cdot C^2\cdot D^6+D^8,$$

$$Y^4=C^8+6\cdot C^4\cdot D^4-4\cdot C^6\cdot D^2-4\cdot C^2\cdot D^6+D^8,$$

$$Z^2=32\cdot C^4\cdot D^4+16\cdot C^6\cdot D^2+16\cdot C^2\cdot D^6.$$

It is immediately verified that the equality (K′) holds. Then, from the equalities (7.51) and (7.53), it follows that:

$$X^4=E^8+6\cdot E^4\cdot F^4+4\cdot E^6\cdot F^2+4\cdot E^2\cdot F^6+F^8,$$

$$Y^4=E^8+38\cdot E^4\cdot F^4-12\cdot E^6\cdot F^2-12\cdot E^2\cdot F^6+F^8,$$

$$Z^2=-32\cdot E^4\cdot F^4+16\cdot E^6\cdot F^2+16\cdot E^2\cdot F^6.$$

Again, it is immediately verified that the equality (K′) holds. □

From Theorem 7.7 we obtain:

Corollary 7.9: *The only solutions of the equation*:

$$x^4+y^4=2\cdot z^2, \tag{L}$$

are given by the equalities:

$$x^2=y^2=k^2. \tag{7.49}$$

***Proof*:** Indeed, since x and y are odd, from equation (L), we obtain the equation:

$$z^4-(x\cdot y)^4=\left(\frac{x^4-y^4}{2}\right)^2,$$

which is an equation of the form (K). So, according to the first equalities from (7.48), the only solutions of the equation (L) are the triplets of the form (k,k,k^2), with $k\in\mathbf{Z}$. □

Corollary 7.10: *The only solutions of the equation*:

$$x^4+6\cdot x^2\cdot y^2+y^4=z^2, \tag{M}$$

are given by the triplets $(0,k,k^2)$ and $(k,0,k^2)$, with $k\in\mathbf{Z}$.

***Proof*:** Indeed, if we replace in equation (L) above, x by x+y and y by x-y, we obtain the equation (M).

***Otherwise*:** Multiplying equation (M) by 16, we obtain:

$$(2\cdot x)^4+6\cdot(2\cdot x)^2\cdot(2\cdot y)^2+(2\cdot y)^4=(4\cdot z)^2, \tag{M$_1$}$$

and if we replace in the equation (M_1) above, $2\cdot x$ with u+v and $2\cdot y$ with u-v, we obtain the equation:

$$u^4+v^4=2\cdot z^2,$$

which is an equation of type (L). Now, Corollary 7.9 completes the proof. □

Corollary 7.11: *The only solutions of the equation*:

$$x^4-6\cdot x^2\cdot y^2+y^4=z^2, \tag{N}$$

are given by the triplets $(0,k,k^2)$ and $(k,0,k^2)$, with $k\in\mathbf{Z}$.

***Proof*:** Indeed, we can assume that:

$$(x,y)=1 \tag{7.50}$$

and rewrite equation (N) as follows:

$$(x^2-y^2)^2-4\cdot x^2\cdot y^2=z^2, \tag{7.51}$$

so:

$$(x^2-y^2-z)\cdot(x^2-y^2+z)=(2\cdot x\cdot y)^2. \tag{7.52}$$

Since, x and y have different parities and z is odd:

$$(x^2-y^2-z,x^2-y^2+z)=2 \tag{7.53}$$

and, from the equality (7.52), it follows that:

$$x^2-y^2-z=2\cdot a^2 \text{ and } x^2-y^2+z=2\cdot b^2, \tag{7.54}$$

with $a, b\in\mathbf{Z}$ and

$$x\cdot y=a\cdot b. \tag{7.55}$$

Then, from the equalities (7.54), we obtain that:

$$x^2-y^2=a^2+b^2 \text{ and } a^4+6\cdot a^2\cdot b^2+b^4=(x^2+y^2)^2.$$

Now we continue as in Corollary 7.10. □

Corollary 7.12: *The only solutions of the equation*:

$$x^4+14\cdot x^2\cdot y^2+y^4=z^2 \tag{O}$$

are given by the triplets: $(0,k,k^2)$ and $(k,0,k^2)$.

Proof**:** Indeed, multiplying the equation by 16 we obtain that:

$$(2\cdot x)^4+14\cdot(2\cdot x)^2\cdot(2\cdot y)^2+(2\cdot y)^4=(4\cdot z)^2, \qquad (7.56)$$

and if we replace in equation (7.56) above, $2\cdot x$ with u+v and $2\cdot y$ with u-v, we obtain the equation:

$$u^4-u^2\cdot v^2+v^4=z^2 \qquad (P)$$

from which, according to Theorem 2.3.3 from (Andreescu & Andrica, 2002, p. 83), it follows that:

$$(u,v,z)\in\{(k,k,k^2), (k,0,k^2)\}.$$

Therefore,

$(x,y,z)=(2\cdot k,0,k^2)$ or $(x,y,z)=(k,k,4\cdot k^2)$.

Corollary 7.13: *The only solutions of the equation*:

$$3\cdot x^4+10\cdot x^2\cdot y^2+3\cdot y^4=z^2 \qquad (Q)$$

are given by the triplets: $(k,k,4\cdot k^2)$.

Proof**:** Indeed, we write the equation like this:

$$(3\cdot x^2+y^2)\cdot(x^2+3\cdot y^2)=z^2 \qquad (Q_1)$$

and because:

$$(3\cdot x^2+y^2,x^2+3\cdot y^2)=1$$

it follows that:

$$3\cdot x^2+y^2=4\cdot s^2 \text{ and } 3\cdot y^2+x^2=4\cdot t^2. \qquad (7.57)$$

Finally, we obtain that:

$$(x,y,z)=(k,k,4\cdot k^2).$$

Otherwise**:** We replace x with u+v, y with u-v and z with $4\cdot t$, we obtain the equation:

$$u^4+u^2\cdot v^2+v^4=t^2, \qquad (R)$$

from where, according to Theorem 2.3.2 from (Andreescu & Andrica, 2002, p. 81), it follows that:

$$(u,v,t)=(k,0,k^2).$$

Therefore,

$(x,y,z)=(k,k,4\cdot k^2)$.

At the end of this paper, we present the following results, which result from those shown above:

Corollary 7.14**:** *If $X\in\mathcal{M}_n(\mathbf{Z})$, then the following statements hold*:

1) *The triplet (X,X,X^2) is a solution of the equation*:

$$X^4+Y^4=Z^2. \qquad (L')$$

2) *The triplets $(0,X,X^2)$ and $(X,0,X^2)$ are solutions of the equation*:

$$X^4+6\cdot X^2\cdot Y^2+Y^4=Z^2. \qquad (M')$$

3) *The triplets $(0,X,X^2)$ and $(X,0,X^2)$ are solutions of the equation*:

$$X^4-6\cdot X^2\cdot Y^2+Y^4=Z^2. \qquad (N')$$

4) *The triplets $(0,X,X^2)$ and $(X,0,X^2)$ are solutions of the equation*:

$$X^4+14\cdot X^2\cdot Y^2+Y^4=Z^2. \qquad (O')$$

5) *The triplets (X,X,X^2), $(0,X,X^2)$ and $(X,0,X^2)$ are solutions of the equation*:

$$X^4-X^2\cdot Y^2+Y^4=Z^2. \qquad (P')$$

6) *The triplet $(X,X,4\cdot X^2)$ is a solution of the equation*:

$$3\cdot X^4+10\cdot X^2\cdot Y^2+3\cdot Y^4=Z^2. \;\square \qquad (Q')$$

Open problem:

Equations (L′), (M′), (N′), (O′), (P′) and (Q′), above, have other solutions in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$, different from those presented in points 1) – 6) of Corollary 7.14?

6. Findings

Therefore, not only equations of the form (A), (B), (C), (D), (E), and (F) can be transposed into the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$, but also equations of the form (G), (H) , (I), (J), (K), (L), (M), (N), (O), (P), (Q) or (R), have this property. Only, if the first 6 types of equations have solutions in this ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$, not all 10 types of equations in the second category are solvable in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$; some have solutions with nonzero components (for example: (J′), (H'_2), (K′), (L′), (P′), (Q′) and (R′)), others have solutions with components null (for example: (H′), (I′), (N′) and (O′)), and others are not solvable (for example: (G′)).

And in these cases of sovability, each of the solutions determined in Paragraph 5 induces a solution $(X^{(n)},Y^{(n)},Z^{(n)})\in\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})\times\mathcal{M}_n(\mathbf{Z})$, according to the model presented in Paragraph 4 of the paper (Vălcan, 2019).

7. Conclusions

As a general conclusion, we can say that any equation of the form (G), (H), (I), (J), (K), (L), (M), (N), (O), (P), (Q) or (R) can be "*immersed*" in a matrix ring of the type $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$, with $n\in\mathbf{N}^*$, any number, at least equal to 2; only, not all these equations can be solved in the ring of integers $(\mathbf{Z},+,\cdot)$, i.e. not all of them have integer solutions.

The same can be said about the corresponding "*submerged* " equation, i.e. in this matrix ring it is usually quite difficult to determine all solutions and then only certain particular solutions are determined. What's more, sometimes there are no such solutions.

Precisely because of this fact, I proposed, to the attentive reader interested in these matters, an open problem. Of course solving it creates a much more complete image of solving these equations in the ring $(\mathcal{M}_n(\mathbf{Z}),+,\cdot)$.

Of course, this work is also one of the Didactics of Mathematics and is also addressed to pupils, students or teachers attentive and interested in these problems, which we believe we have formed, in this way, a good image about solving these types of equations.

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References

Acu, D., (2010), *Aritmetica si teoria numerelor* [Arithmetic and Number Theory], Sibiu, Editura Universității „*Lucian Blaga*”.

Andreescu, T, .& Andrica, D., (2002), *O introducere in studiul ecuațiilor diofantiene* [An Introduction to the Study of Diofantian Equations], Zalău, Editura GIL.

Cucurezeanu, I., (2005), *Ecuaţii în numere întregi (Equations in integer numbers)*, București, Editura Aramis Print.

Elkies, D. N., (1988), *On $A^4 + B^4 + C^4 = D^4$*, in *Math. of Comp.* 51 (Oct. 1988), pp. 825-835.

Frye, R. E., (1988), *Finding, $95800^4+217519^4+414560^4=422481^4$ on the Connection Machine*, in *Supercomputing '88: Proceedings of the 1988 ACM / IEEE Conference on Supercomputing, Vol. II, Science and Applications*, pp. 106-116.

Vălcan, D., (2017), *Formarea inițială a profesorilor de matematică la universitatea „babeş-bolyai", nivel licență, Curriculumul Național matematic preuniversitar în perioada 2000 - 2016*, Vol. I: *De la obiective / competențe la planuri - cadru de învățământ* [INITIAL TRAINING OF MATHEMATICS TEACHERS IN "BABEȘ-BOLYAI" UNIVERSITY, LICENSE LEVEL, National Pre - University Mathematical Curriculum in the Period 2000 - 2016, Vol. I: From Objectives / Competencies to curriculum framework plans], Cluj-Napoca, Editura Casa Cărții de Știință.

Vălcan, D., (2019), *From Diofantian Equations to Matricial Equations (I) - Equations and Pythagorean Matrices*, in *Journal of Education & Social Policy*, Vol. 6 (1), 2019, pp. 60-83.

Vălcan, D., (2022), From diofantian equations to matricial equations (ii) - generalizations of the pythagorean equation –, in *Education, Reflection, Development – ERD 2021*, Published by the Future Academy / EPES, Volume 2, edit.: Albulescu, I., Stan, N.-C., 2022, pp. 627-641.

http://mathworld.wolfram.com/topics/Algebra.html