10.24193/ed21.2021.21.11

How can we illustrate the usefulness of the differential calculus course taught in a technical university

Ștefania Constantinescu, Rodica-Mihaela Dăneț, Marian-Valentin Popescu

Educatia 21 Journal 21 (2021)

Educatia 21 Journal, (21) 2021, Art. 11
doi: 10.24193/ed21.2021.21.11
Research article

How can we illustrate the usefulness of the differential calculus course taught in a technical university

Ștefania Constantinescu [a*], Rodica-Mihaela Dăneț [b], Marian-Valentin Popescu [c]

[a] *Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bd. Lacul Tei nr. 122 - 124, cod 020396, Sector 2, Bucharest, Romania*
[b] *Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bd. Lacul Tei nr. 122 - 124, cod 020396, Sector 2, Bucharest, Romania*
[c] *Department of Metal Construction, Management and Engineering Graphics, and Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, Bd. Lacul Tei nr. 122 - 124, cod 020396, Sector 2, Bucharest, Romania*

*Corresponding author: *stefania.constantinescu@utcb.ro*

Abstract

Working in a challenging academic environment as mathematics professors for the Technical University of Civil Engineering Bucharest, we thought to find a teaching strategy that, in addition to the required standard, but also has the standard of attractiveness and accessibility. Why? This is due to the fact that our students have a very nonhomogeneous level of knowledge and logical-mathematical skills. That's how we came up with the idea of organizing each lesson in our courses in a gradual way, and completing them with an informal part. On the other hand, we set out to collaborate with professors working in the departments of other natural sciences or in engineering departments, enriching the mathematics courses with technical applications. This has led to write several didactic works, which also include such applications. Proving the usefulness of this courses, our new work offers two such technical applications for the mathematical analysis course, taught in the first semester and dedicated to the differential calculus of functions having one or several variables. More precisely, we present in a gradual way, two applications solved by using the mathematical modelling: a problem belonging to electricity and then, the cruising speed problem. The gradual presentation begins, for each of these problems, with the necessary notions (organized in the form of two dictionaries, for Math and for Physics, respectively), continues with the statement of the problem, with the solution methodology, and finally, with the solution itself. Our presentation will provide students with a model of logical (mathematical) approach, useful to them in the courses of other natural sciences and of engineering disciplines that they will study later. In addition, it will prove them why mathematics is a fundamental discipline for the engineering education.

Keywords:
Differential calculus; Lagrange theorem; study of the variation of functions; mathematical modelling; secondorder linear nonhomogeneous differential equation with constant coefficients; Cauchy problem; equation of motion; electrical current; current with

Zusammenfasung

Wir haben reiche Erfahrung als Mathematikprofessoren an der Technischen Universität für Bauingenieurwesen Bukarest. In diesem akademischen Umfeld haben wir uns überlegt, eine Lehrstrategie zu finden, die neben dem geforderten Standard auch den Anspruch an Attraktivität und Zugänglichkeit hat. Warum? Dies liegt daran, dass unsere Studierenden einen sehr inhomogenen Wissensstand und logisch-mathematische Fähigkeiten haben. So kamen wir auf die Idee, jede Unterrichtsstunde in unseren Kursen stufenweise zu gestalten und mit einem informellen Teil zu ergänzen. Andererseits haben wir uns vorgenommen, mit Professoren aus anderen naturwissenschaftlichen oder ingenieurwissenschaftlichen Abteilungen zusammenzuarbeiten, um das Mathematikstudium um technische Anwendungen zu bereichern. Dies hat dazu geführt, dass mehrere didaktische Arbeiten verfasst wurden, die auch solche Anwendungen enthalten. Als Beweis für die Nützlichkeit dieser Kurse bietet unsere neue Arbeit zwei solcher technischen Anwendungen für den im ersten Semester gelehrten mathematischen Analysis-Kurs, der sich der Differentialrechnung von Funktionen mit einer oder mehreren Variablen widmet. Genauer gesagt stellen wir nach und nach zwei Anwendungen vor, die mit Hilfe der mathematischen Modellierung gelöst wurden: ein Problem der Elektrizität und dann das Problem der Reisegeschwindigkeit. Die schrittweise Darstellung beginnt für jedes dieser Probleme mit den notwendigen Begriffen (organisiert in Form von zwei Wörterbüchern für Mathematik bzw. für Physik), geht weiter mit der Problemstellung, mit der Lösungsmethodik und schließlich mit die Lösung selbst. Unsere Präsentation wird den Studierenden ein Modell des logischen (mathematischen) Ansatzes an die Hand geben, das ihnen in den Studiengängen anderer Naturwissenschaften und Ingenieurwissenschaften, die sie später studieren werden, nützlich ist. Darüber hinaus wird ihnen gezeigt, warum Mathematik eine grundlegende Disziplin für die Ingenieurausbildung ist.

Schlüsselworte:
Differentialrechnung; Lagrange-Theorem; Studium der Variation von Funktionen; mathematische Modellierung; lineare inhomogene Differentialgleichung zweiter Ordnung mit konstanten Koeffizienten; Cauchy-Problem; Bewegungsgleichung; elektrischer Strom; Strom

1. Introduction

The authors of this paper strongly believe that *teaching mathematics in a technical university* is a didactic challenge. This is because, in addition to the rigor of the traditional way of teaching, the teacher must also be concerned with increasing the *accessibility* and *attractiveness* of the presentation. To this end, the teacher should organize his presentation gradually, possibly adding an informal part, pointing out very briefly, if possible, the place of the subject in the history and philosophy of mathematics.

At the same time, the same professor who teaches mathematics in a technical university is put in the situation of finding a common language with the professors who teach the other fundamental disciplines and the engineering disciplines, the main goal being the mathematical modelling of some technical phenomena. Obviously, a step in establishing this common language is to find the most suitable technical applications and use the usual terminology and notations in the discipline applied in solving that application.

The person who teaches a mathematics course included in the technical higher education program must be aware that he must introduce his students to the technical disciplines that they will study later. He must also convince the students that everything they learn in mathematics will be useful in understanding other subjects.

It would be best for, for example, students in technical education to hear at the Course of the Mathematical Analysis about certain problems in Physics or in Mechanics and not vice versa, that is, to hear later that in solving such problems they need certain algorithms that were not taught in the math course.

We will exemplify this, showing how two problems in physics (from the “Electricity” chapter and the “Mechanics” chapter, respectively) can be solved:

1) A problem of electricity; 2) Cruise Speed Problem.

2. Theoretical foundation

Since we want this paper to be autonomous, before formulating (in the section “3. Research Methodology”) the two problems mentioned at the end of the previous section, we mention everything necessary to understand and solve these problems. In this section we introduce the *notions of mathematics* and *physics* and subsequently, in the next section, the *algorithms* we will use.

2.1 The problem of electricity

For the statement of the problem, see Section 3.1.

2.1.A Dictionary of mathematics

We mention the following definitions:

Definition 1. If $J \subseteq \mathbb{R}$ is an interval, we say that a function $f : J \to \mathbb{R}$ is *strictly increasing* (*strictly decreasing*, respectively) on J, if for all $x_1, x_2 \in J$, with $x_1 < x_2$, the following inequality is valid: $f(x_1) < f(x_2)$ ($f(x_1) > f(x_2)$, respectively).

Definition 2. If $J \subseteq \mathbb{R}$ is an interval, we say that a function $f : J \to \mathbb{R}$ is *increasing* (*decreasing*, respectively) on J if for all $x_1, x_2 \in J$ with $x_1 \le x_2$, the following inequality is valid: $f(x_1) \le f(x_2)$ ($f(x_1) \ge f(x_2)$, respectively).

Remark. In the study of the variation of a function, study made for the purpose of the graphical representation of this function, a statement is used, which is a consequence of Lagrange’s Mean Value Theorem. First we mention the statement of Lagrange’s Mean Value Theorem.

Lagrange’s Mean Value Theorem (also known as **First Mean Value Theorem).** *Let* $f : [a,b] \to \mathbb{R}$ *be a Rolle function, that is, a function having the following properties:*

1) f *is continuous on the closed interval* $[a,b]$*;*

2) f *is differentiable on the open interval* (a,b).

Then there is at least one point $c \in (a,b)$, *such that* $f'(c) = \dfrac{f(b) - f(a)}{b-a}$ *or, equivalently,* $f(b) - f(a) = f'(c)(b-a)$.

Lagrange's Mean Value Theorem has several consequences, used in the study of the variation of functions. Of these, the following shows the connection between the *sign of the derivative* f' of a differentiable function f on an interval $J \subseteq \mathbb{R}$ and the *monotony of* f. (Remind that a function $f : J \to \mathbb{R}$ is called *monotone on* J, if it is either increasing or decreasing on J.)

Corollary of Lagrange's Mean Value Theorem. *Let* $f : J \to \mathbb{R}$ *be a differentiable function on an interval* $J \subseteq \mathbb{R}$.

a) *If* $f' \geq 0$ *on* J *(that is,* $f'(x) \geq 0$, *for any* $x \in J$*), then* f *is increasing on* J

b) *If* $f' \leq 0$ *on* J *(that is,* $f'(x) \leq 0$, *for any* $x \in J$*), then* f *is decreasing on* J

a') *If* $f' > 0$ *on* J *(that is,* $f'(x) > 0$, *for any* $x \in J$*), then* f *strictly increasing on* J;

b') *If* $f' < 0$ *on* J *(that is,* $f'(x) < 0$, *for any* $x \in J$*), then* f *is strictly decreasing on* J.

Remark. The following two tables synthesize a') and b') from the above statement, for $J = [a, b]$.

Case a')

x	a		b
$f'(x)$	$+ + + + + + + +$		
$f(x)$	$f(a)$	$\nearrow$	$f(b)$

Case b')

x	a		b
$f'(x)$	$- - - - - - - -$		
$f(x)$	$f(a)$	$\searrow$	$f(b)$

The statement also applies to $J = (a, b]$ or $J = [a, b)$ or $J = (a, b)$ and also, for $J = (-\infty, +\infty)$ or $J = (a, +\infty)$ or $J = (-\infty, a)$, with $a, b \in \mathbb{R}$.

2.1.B Dictionary of physics

The following notions appear in any elementary book on electricity, for example, in (Tamm, I., 1952).

- An *accumulator battery* is a rechargeable source of direct current, consisting of elements that store the electricity, by using chemical principles. Its operation is based on the appearance of an electromotive voltage created on chemical bases, obtained by combining in electrode-electrolyte combinations of different materials from an electrochemical point of view.

- The *electromotive voltage* is a physical quantity equal to the electrical voltage (see the definition below) at the terminals of an open circuit electric generator (that is, the positive terminal and the negative terminal are not connected and there is no electrical circuit). Usually, the electromotive voltage is denoted by e.

- The *electric potential* at a point in an electric field is a scalar field-type physical quantity that characterizes the electric field at that point; is defined as the ratio of the electrical work done to move a positive electric charge from infinity to that point and the value of that charge. The potential at infinity is chosen to be zero. Thus the electric potential for a point charge decreases with distance.

- The *electrical voltage between two points* of an electrical circuit is equal to the potential difference between the two points and represents the ratio between the electrical work done to move a positive charge between the two points and the size of that charge. Usually, the electrical voltage is denoted by U.

- The *intensity* of the electric current is a fundamental scalar physical quantity that measures the strength of the effects of the electric current. We refer to the *thermal effect* (Joule), the *chemical effect* (electrolysis) and the *magnetic effect*. Usually, the intensity of the electric current is denoted by i.

- The *simple electrical circuit* consists of at least one voltage source, connecting conductors (field guides) and a consumer.

- *Electrical resistance* is defined by the ratio between the voltage applied to its ends and the intensity of the current flowing through it. Physically, this means the ability of a conductor to resist the passage of electric current through it. The unit of measurement of electrical resistance, in SI, is the *ohm* denoted by Ω.
- The *internal resistance* is the resistance inside the source. Usually, it is denoted by r.
- *External resistance* is the resistance of what does not belong to the source. Usually, it is denoted by R.

In the following, we will apply *Ohm's Law* or the *law of electrical conduction*, which establishes the connection between the intensity i of the electric current, the applied electric voltage U and the total resistance R_t in the circuit, namely $I = \dfrac{U}{R_t}$.

2.2 The cruising speed problem

For the statement of the problem, see Section 3.2.

2.2.A Dictionary of mathematics

Definition 3. A *first-order linear differential equation, with constant coefficients* is an equation of form $a_1y'(x)+a_0y(x)=f(x)$, where $a_1, a_0 \in \mathbb{R}$, $a_0 \neq 0$ and $x \in J$ (J is an interval of the real axis) and $f : J \to \mathbb{R}$ is a continuous function. For simplification, we denote $a_1 = a$ and $a_0 = b$. With these notations, the above equation becomes:

$$a \cdot y'(x)+b \cdot y(x)= f(x), \tag{1}$$

where $a,b \in \mathbb{R}$, $a \neq 0$ and $x \in J$.

Definition 4. The first-order linear differential equation (1) is called:

1) a *homogeneous equation*, if the right member $f(x)$ of the equation satisfies the condition: $f(x)=0$, for any $x \in J$.

2) a *nonhomogeneous equation*, on the contrary, that is, if the right member $f(x)$ of the equation satisfies the condition: there exists $x \in J$, with $f(x) \neq 0$.

Remark. According to „Definition 4. 1)", we deduce that a *first-order homogeneous linear differential equation* is of the form :

$$ay'(x)+by(x)=0. \tag{2}$$

It is shown that the form of the *general solution of the nonhomogeneous linear equation* (1) willl be $y_o = y(x,C)$, with $C \in \mathbb{R}$, that is, it will depend on a real constant C. To determine a *particular solution* of the nonhomogeneous linear equation (1) it is necessary to particularize the constant C. To this aim, it is necessary to know an *initial condition* that the nonhomogeneous linear differential equation (2) must satisfy. Usually this "initial condition" imposes on the unknown function $y = y(x)$ the condition that $y(x_0) = y_0$, where $x_0 \in J$ and y_0 are two known real numbers.

Definition 5. It is called a *Cauchy problem* attached to the first-order nonhomogeneous linear differential equation (1), that is, to the equation $ay'(x)+by(x)= f(x),\ x \in J,$ a problem that requires determining a particular solution y_p of this equation, such that y_p checks the initial condition $y_p(x_0)= y_0$, where $x_0 \in J$ and $y_0 \in \mathbb{R}$ are two known numbers.

Remark. In other words, if $y_n(x,C)$ is the *general solution* of the differential equation (1), a *Cauchy problem* attached to this equation by the initial condition $y_p(x_0)= y_0$, requires determination of the constant C, so that the graph of the particular solution y_p of the equation (1) pass through the point $M(x_0, y_0)$ from the xOy plan.

Next, we recall how to solve the first-order ***nonhomogeneous*** linear differential equation, with constant coefficients (1).

Step 1. We start with *solving the first-order* ***homogeneous*** *linear differential equation* (2) attached to the equation (1) (by omitting the right member $f(x)$ of the latter). So we have to solve the equation (2) $(ay'+by=0)$ with $a,b\in\mathbb{R}$, $a\neq 0$.

Replacing y' by $\frac{dy}{dx}$, the equation (2) is written equivalent:

$$a\frac{dy}{dx}+by=0. \tag{3}$$

We observe that, in the first-order differential equation (3) the "variables" can be "separated", that is, (3) can be equivalently transformed into an equality, so that in the left member it "appears" only y and in the one on the right, only x. (Therefore, it is said that equation (3) "has *separable variables*".) Thus, from (3), it follows that:

$$\frac{dy}{dx}=-\frac{b}{a}y. \tag{4}$$

To "separate the variables" in this equation, we multiply by dx and we divide by y; but then we will have to impose the condition $y\neq 0$. We have two cases to study.

Case A): $y\neq 0$. Then from (4), it follows $\frac{dy}{y}=-\frac{b}{a}dx$. Now we integrate in both members and we logarithmically note the integration constant (which appears when calculating the primitives), in the form $\ln k$ with $k>0$. It follows:

$$\int\frac{dy}{y}=-\frac{b}{a}\int dx\Rightarrow\ln|y|=-\frac{b}{a}x+\ln k$$, with $k>0\Rightarrow$

$$\ln|y|-\ln k=-\frac{b}{a}x$$, with $k>0\Rightarrow$

$$\ln\frac{|y|}{k}=-\frac{b}{a}x\ (k>0)\Rightarrow|y|=k\cdot e^{-\frac{b}{a}x}\ (k>0)$$

$$\Rightarrow y=\pm k\cdot e^{-\frac{b}{a}x}$$, with $k>0$.

We denote $\pm k=C$ and because $k>0$, it follows $C\in\mathbb{R}^*$ (that is, $C\neq 0$). So, in this case, the general solution y_o of the first-order homogeneous linear differential equation (2) is:

$$y_o=C\cdot e^{-\frac{b}{a}x}, \tag{5}$$

where $C\in\mathbb{R}^*$.

Case B): $y=0$. Then $y'=0$ and so $y=0$ checks the equation (2), that is, the equation $ay'+by=0$. So we complete the solution (5) of this equation (2) with $y=0$.

We wonder if $y=0$ is a *particular solution* or, respectively, a *singular solution* for the equation (2). We remind that:

1) It is called a *particular solution* of the differential equation (2) a solution of this equation that can be obtained from the general solution (5) by particularizing the real constant C.

2) It is called a *singular solution* of the differential equation (2) a solution that cannot be obtained from the general solution (5) by no particularization of the real constant C.

We notice that if we complete (5) with the value $C=0$ we obtain:

$$y_o=C\cdot e^{-\frac{b}{a}x}, \tag{6}$$

where $C\in\mathbb{R}$. For $C=0$ in (6), it follows $y=0$, which is exactly the additional solution of the equation (2) discussed in the "*Case B)*". Therefore $y=0$ it follows from y_o by particularizing the constant C in (6). In other words, $y=0$ is a particular solution of the equation (2).

Step 2. We will look for a *particular solution* for the *first-order* ***nonhomogeneous*** *linear differential equation* (1), that is, for the equation

$a \cdot y' + b \cdot y = f(x)$. For this, we will use the ***Method of variation of constants*** (***Euler-Lagrange Method***). (Actually, in this case, this method is more correctly called the ***Method of variation of constant***, because there exists a single constant.) Since from (6), the *first-order **homogeneous** linear differential equation* (2) ($ay' + by = 0, x \in J, a, b \in \mathbb{R}$ and $a \neq 0$) has the solution $y_o = C \cdot e^{-\frac{b}{a}x}, C \in \mathbb{R}$, we will search for the ***nonhomogeneous*** equation (1) a particular solution y_p, having a similar form with y_o, but replacing the constant C with a function $C(x)$, unknown for instant. Hence:

$$y_p = C(x) \cdot e^{-\frac{b}{a}x}. \tag{7}$$

But

$$y_p' = C'(x) \cdot e^{-\frac{b}{a}x} + C(x)\left(-\frac{b}{a}\right)e^{-\frac{b}{a}x}. \tag{8}$$

We introduce (7) and (8) in the equation (2). It follows:

$$a \cdot y_p' + b \cdot y_p = f(x) \Rightarrow$$

$$a \cdot C'(x)e^{-\frac{b}{a}x} - b \cdot C(x)e^{-\frac{b}{a}x} + b \cdot C(x)e^{-\frac{b}{a}x} = f(x)$$

$$\Rightarrow C'(x) = \frac{f(x)}{a \cdot e^{-\frac{b}{a}x}} \Rightarrow C'(x) = \frac{1}{a} f(x) \cdot e^{\frac{b}{a}x}$$

By integrating, we get $C(x) = \frac{1}{a}\int f(x) \cdot e^{\frac{b}{a}x} dx$. Replacing this function in (7), we get:

$$y_p(x) = \left(\frac{1}{a}\int f(x) \cdot e^{\frac{b}{a}x} dx\right) \cdot e^{-\frac{b}{a}x}. \tag{9}$$

Step 3. The general solution y_n of the ***nonhomogeneous*** linear differential equation (1) $ay' + by = f(x),\ x \in J$ is:

$$y_n(x) = y_o(x) + y_p(x), x \in J, \tag{10}$$

where:

1) $y_o(x)$ is the general solution of the ***homogeneous*** linear differential equation (2) attached to the equation (1). It was determined at "Step 1.", see (6);

2) $y_p(x)$ is a particular solution of the ***nonhomogeneous*** linear differential equation (1). It was determined at "Step 2. ", see (9).

From (6) (in which we denoted C by D), (9) and (10) we deduce:

$$y_n(x) = D \cdot e^{-\frac{b}{a}x} + \left(-\frac{1}{a}\int f(x) \cdot e^{\frac{b}{a}x} dx\right)e^{-\frac{b}{a}x}$$

and hence

$$y_n(x) = e^{-\frac{b}{a}x}\left(D - \frac{1}{a}\int f(x) \cdot e^{\frac{b}{a}x} dx\right), x \in J, \tag{11}$$

where $D \in \mathbb{R}$ is the "constant of integration".

2.2.B Dictionary of physics

Actually, the notions of physics that we will remind, are taught to high school students, in the chapter 'Dynamics', which belongs to the classical mechanics. In this chapter, the relationship between the forces acting on a physical body and its movement is studied.

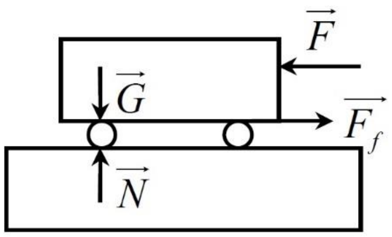
- *The traction force* (or, equivalent, the *tractive force*) $\vec{F}$ of a moving car is produced by its engine. By overcoming the resistances that oppose this movement, the traction force of the car causes it to move.

Convention. In the following, we will say, for example, the "*traction force*", both to the vector $\vec{F}$, and of the size (length, magnitude, modulus) F of this vector. We will do the same with all the forces that appear. The convention is justified, because we will discuss, for example, about the *speed* v of the car, but in a force diagram, we will represent the vector $\vec{v}$ and we will call it, also *speed.*

- *Friction force. Coefficient of friction. Normal force* (*reaction*). The *friction force* (also called the *kinetic friction force*) $\vec{F_f}$ at the contact surface between two bodies is the tangential component of the support force that one of the bodies exercises on the second body, see (Răduleţ, R. et. al, 1957-1958). While driving a car, the frictional force opposes the movement, slowing it down. The history on studies on the friction force (studies that were initially experimental) showed that friction is a complex phenomenon, which can depend on many parameters.

The following figure represents the force diagram in case of a car movement.

Figure 1. Diagram of forces



These forces are: $\vec{F}$, $\vec{F_f}$, $\vec{G}$ (*weight*), $\vec{N}$ (*normal force* or, equivalent, *normal reaction*). The last two forces are the *interaction forces* between the two surfaces (one of these surfaces being the road on which the car travels). They prevent the two bodies in contact from passing through each other.

It is shown that the friction force $\vec{F_f}$ is generally given by the equality $\vec{F_f} = \mu \vec{N}$, where μ is the *coefficient of friction*. So the coefficient μ of friction is the proportionality factor between the sizes of $\vec{F}$ and $\vec{N}$.

Comment. We will see, in “Cruising Speed Problem”, that we are interested in writing the equation of motion for speed control of the mechanical system, represented by a moving car, whose engine develops a traction force $\vec{F}$. For this, we will make a mathematical modelling of the system, by using differential equations, the analysis of the system being in the time domain. Basically, we will make a description of the system, using mathematical notions and algorithms. Following the mathematical modelling will result a mathematical model, obtained based on working hypotheses, called *simplifying hypotheses.*

In the matter of cruising speed, which we want to “model”, *simplifying assumptions* will be made about the friction force $\vec{F_f}$, precisely because, as mentioned above, it can depend on many parameters, sometimes too many to allow “modelling”. A presentation of these *simplifying hypotheses* can be found in (Popova, E. et. al, 2015). If we look at the history of physics, looking for the moments when these hypotheses were outlined, the best known is the “Coulomb moment”. But before Charles-Augustine Coulomb (1736 - 1806), we must mention (even if his statements did not influence the science and engineering of his time), Leonardo Da Vinci (1452-1519), in whose famous notebooks can be found the main laws on the dry friction, see, for example, [Dowson, 1979]. Among these:

(F1) The friction force does not depend on the area of the contact surface.

About a century and a half after Da Vinci, Guillaume Amontons was born (1663-1705), the author of the first study published (in 1699) on friction, (Amontons, 1699). He formulated 4 laws of friction. Among these:

(F2) The friction force does not depend on the speed;

(F3) The friction force is proportional to the normal force (hypothesis known as *Amontons' Law*).

Amontons' work greatly influenced engineering practice, see (Popova, E. et. al, 2015). In the eighteenth century, Coulomb, mentioned above, would have led lengthy experimental studies on friction, a phenomenon that had proved to be complex. Coulomb's name is emblematic of the evolution of Physics and Engineering in France in that century. Coulomb's first seminal work was his memoir “*Théorie des machines simple*” (“*Theory of Simple Machines*”). The *dry friction* was known as the *Coulomb rubbing*. A study of this type of friction is performed in (Popov, 2010). Coulomb confirmed Amontons' law, but also studied the dependencies of friction on other parameters, even if these dependencies are weak. If we think intuitively, among these parameters, in the case of moving a car there is also the speed $\vec{v}$. Indeed, when $\vec{v}$ increases, in addition to road friction, there is also air friction, which makes the friction force no longer independent of $\vec{v}$, contradicting what is stated in **(F2)**. (Of course, it is possible that this *simplifying hypothesis* may be taken into account in the case of low speeds.) A study of the dependence relationship between kinetic friction and velocity as well as a history of this problem also appear in (Braun, O. ; Peyrard, M., 2011).

In the mathematical modelling that we will use in this paper to determine the cruising speed, we will consider that the **friction force** F_f **depends linearly on the speed** v of the car. However, it is normal to also appeal to a law of Newton, as it is well known that *Newton's Laws*, also called the *Fundamental Principles of Mechanics*, are three laws that establish a connection between the movement of a body and the forces acting on it. Isaac Newton (1642-1727), based on the studies of his predecessor Galileo Galilei (1564-1642), wrote in 1687 a monumental work, fundamental to classical mechanics ("*Philosophiae Naturalis Principia Mathematica*"), in which he formulated these laws. In our paper we will use the **Second Principle of Dynamics** (also called **Newton's Second Law**). To state this law, we consider a force $\vec{F}$ that acts on a body, and imprints on it an acceleration $\vec{a}$ collinear with $\vec{F}$, and whose size $\vec{a}$ is proportional to the size F of $\vec{F}$ and inversely proportional to the mass m of the body. The relationship between $\vec{F}$, m and $\vec{a}$ is

$$\vec{F} = m\vec{a}.$$

We mention that, working in the time domain *t,* the acceleration $a(t)$ of the moving body shows how fast the velocity $v(t)$ changes:

$$a(t) = \frac{dv}{dt} \text{ or,}$$

with the notation used in mechanics,

$$\vec{a} = \dot{\vec{v}}.$$

3. Research methodology

3.1 The problem of electricity

We consider 72 battery elements with the same electromotive voltage e and each with internal resistance r and external resistance R.

Find the most efficient method of grouping the 72 elements in α series of β elements each, grouped in parallel so as to obtain a current of maximum intensity.

The ***methodology for solving*** the problem begins with its wording in the language of the Mathematical Analysis. For this, we will build a function f in the variable β and study its monotony, by applying the consequence of the Lagrange's theorem from the differential calculus of a function having only a variable, see 2.2.A (the **Dictionary of mathematics**). Solving will continue by making estimates to find the most effective method of grouping the 72 elements, in the meaning of the problem.

3.2 The cruising speed problem

In the Oxford English Dictionary (OED), the *cruising speed* is defined as "a speed for a particular vehicle, ship, or aircraft, usually somewhat below maximum, that is comfortable and economical".

Expressing ourselves in an informal language, we are interested in the speed at which we can constantly drive a car, for optimal consumption and minimal engine wear, an important benefit being that we can drive on long roads without too much a lot of fatigue. In this sense, excluding possible engine wear, cruising speed is a kind of economical speed, or rather, a kind of optimum between speed and fuel consumption.

Comments

a.) If we refer to the **technical significance** of the problem, at any course of economic driving, we learn that engine manufacturers believe that the cruising speed is somewhere at about 75% of the maximum speed that the car can reach.

Obviously, it is important for any driver (especially for the truckers traveling thousands of kilometers per day), to identify and use cruising speed. Now, many modern cars, even cheaper models, are equipped with a cruise system. In fact, it is a cruise speed control system and has appeared for a long time, before many other systems that facilitate the control of the car. With the first such cruise control systems, American cars have been equipped for many years, because they usually run very long distances. Speaking more technically, but only informally, cruising speed control is done with a servo-device that connects to the vehicle's on-board computer and adjusts the throttle opening.

But in this didactic work, we are not interested in these technical aspects. Our concern is to make a **mathematical modelling** of this phenomenon.

b.) It is obvious that the **theoretical study of the problem of determining the cruising speed,** on which the control systems are based, is of interest.

As with the previous problem, the **methodology for solving** this problem begins with the formulation of the problem in the language of Mathematical Analysis, following the application of the theory of differential calculus of functions having a single variable. Using the second principle of dynamics (Newton's principle) we arrive at a first-order nonhomogeneous linear differential equation, the solution of which will give the cruising speed $v(t)$ of the car at time t. In fact, the differential equation that gives this velocity is accompanied by an initial condition ($v(0)=0$). Thus, we have to solve a Cauchy problem. Since $v(t)=x'(t)$, we can then determine the distance $x(t)$ traveled by the car up to time t.

4. Results

4.1 The problem of electricity

We remind again the statement of this problem.

We consider 72 battery elements with a same electromotive voltage e and each with internal resistance r and external resistance R.

Find the most effective method of grouping the 72 elements in α series of β elements each, grouped in parallel, so as to obtain the maximum intensity of the current.

Solving the electricity problem

To each series of β elements, we have the electromotive tension βe and the internal resistance βr. For all the battery, when we put in parallel the ones α series, the electromotive force is $(V=)\beta e$ and the internal resistance is $\frac{\beta r}{\alpha}$. This leads to a total resistance equal to $R+\frac{\beta r}{\alpha}$. The electric current intensity will be $i=\frac{\beta e}{R+\frac{\beta r}{\alpha}}$; since $\alpha=\frac{72}{\beta}$, it follows $i=\frac{\beta e}{R+\frac{\beta^2 r}{72}}$. So we obained the electric current intensity i as a function of the variable β. We aim to determine the monotony of the function i. We will note $i=f$ and $\beta=x$. We have the function $f(x)=\frac{ex}{R+\frac{rx^2}{72}}$. We aim to apply the consequence of the Lagrange theorem, mentioned in Section 2.1.A. For this, we calculate the derivative of the function f. It follows:

$$f'(x)=e\frac{R+\frac{rx^2}{72}-\frac{rx^2}{36}}{\left(R+\frac{rx^2}{72}\right)^2}\Rightarrow f'(x)=e\frac{R-\frac{rx^2}{72}}{\left(R+\frac{rx^2}{72}\right)^2}.$$

We remark that:

$f'(x)>0$ on the interval $\left(0,\sqrt{\frac{72R}{r}}\right)$, and

$f'(x)<0$ on the interval $\left(\sqrt{\frac{72R}{r}},+\infty\right)$.

According to the consequence of the Lagrange theorem, it follows:

x	0	$\sqrt{\frac{72R}{r}}$	$+\infty$
$f'(x)$	+ + + + + + +	0	− − − − − − −
$f(x)$	□	$f\left(\sqrt{\frac{72R}{r}}\right)$	□

To solve our problem, we are looking for two consecutive divisors of 72, between which the real number $\sqrt{\frac{72R}{2}}$ is found and, next, we will determine for which of these divisors the higher intensity is obtained.

Particular cases:

1) $r=1\Omega,\ R=3\Omega$

2) $r=1\Omega,\ R=15\Omega$

3) $r=1\Omega,\ R=100\Omega$

4) $r=1\Omega,\ R=2\Omega$

5) $r=1\Omega,\ R=10\Omega$

We will analyze cases 3) and 5) the other cases leaving them as exercises.

Case 3. For $R=100$, it follows that $\sqrt{\frac{72\cdot 100}{1}}>72$. From the table above we deduce that the function f increasing on the interval $(0,72)$, so we take $\beta=72$ and $\alpha=1$.

Case 5. For $R=10$, it follows that $\sqrt{\frac{72\cdot 10}{1}}\cong 26,83$, which is between the divisors 24 and 36 of 72.

Remark. The most effective choice is $\alpha=3, \beta=24$ ($f(24)>f(36)$).

Ω

4.2 The cruising speed problem

We remind again the statement of the problem of "cruising speed", but we express ourselves in a more applied, more precise language.

We are interested in determining what is the size v of the (ideal) cruising speed, which a driver must keep on the highway, in order to have the lowest possible consumption, but running at a normal speed for a highway. Basically, we will work in the field of time t, finding, in certain hypotheses, the *expression of the size* $v(t)$ *of the speed at time* t and, consequently, the *equation corresponding to the motion of the car*. We assume that its engine develops traction force $\vec{F}$.

In this section we will apply the differential calculus of a function with one variable, to show how the cruise control system can be modelled.

In addition to the traction force $\vec{F}$, in the drawing from 2.2.B, there is also the friction force $\vec{F_f}$ that appears during the movement and opposes the movement. The size F_f of the friction force depends on the sides G and N of the two forces of interaction, $\vec{G}$ (the *weight of the car*) and $\vec{N}$ (the *normal force* or *normal reaction*) between the two surfaces in contact, the pair of forces (coming from gravitational acceleration), which prevents the bodies from passing through each other. We mention that in science and engineering, the weight of an object is related to the force acting on the object, either due to gravity or a reaction force that holds it in place.

We also mentioned that the size F_f of the friction force is generally given by equality $F_f=\mu N$, where μ is the *coefficient of friction.*

Solving the cruising speed problem

Let's start with the ***Problem data***.

We work in the field of time t, and we consider the size of the forces that appear. We will refer to each of these sizes with the same name as the force of which it is associated.

We assume that we know:

a) The motor-developed force, denoted by F

b) The mass m of the car;

c) The friction coefficient k_f with the road on which the car is travelling; this coefficient is constant and the road friction force, $F_f=F_f(t)$, it is directly proportional to the speed v, the proportionality factor being k_f, that is, $F_f(t)=k_f v(t)$, at the time t.

We start with a short comment, see 2.2.B, concerning "***The second principle of dynamics*** (***Newton's principle***)", applied for an object which has a constant mass and is in motion (for the *cruising speed problem,* the object is a car in motion). This ***principle*** (which is a law of classical mechanics that describes a relation between the motion of the object and the forces acting on it) states that: *the size* R of the resultant of these forces is equal to $m\cdot a$, where m is the mass of the object and a is its acceleration.

Choosing an origin (the starting point of the car) and denoting by $x(t)$, the distance (from this origin) where the car is after a while t, the speed $v(t)$ and the acceleration $a(t)$ are given by:

$$v(t)=x'(t)$$

and:

$$a(t)=v'(t)=x''(t), \quad (12)$$

respectively.

We want to set the *differential equation of the motion of the car*, that is, to determine the function $x=x(t)$, $t\geq 0$. We will use the second principle of dynamics. Since F_t is opposed to force F, the resultant of the forces acting on the car is $R(t)=F(t)-F_f(t)$. It follows:

$$R(t)=F(t)-k_f v(t)=F(t)-k_f\cdot x'(t) \quad (13)$$

But, from the second principle of dynamics, it follows:

$$R(t)=ma(t)\overset{(12)}{=}mx''(t),$$

hence

$$R(t)=mx''(t) \quad (14)$$

From (13) and (14), it follows:

$$F(t)-k_f x'(t)=mx''(t). \quad (15)$$

We have to solve a Cauchy problem as we obviously have initial conditions given by the values of the functions $x(t)$ and $v(t)=x'(t)$ at the initial time $t=0$. More precisely:

$$x(0)=0 \text{ and } v(0)=0 \quad (16)$$

or equivalently $x(0)=0,\ x'(0)=0.$

The equation (15) is a ***second-order*** *nonhomogeneous linear differential equation, with constant coefficients*, but in incomplete form, because the term $x(t)$ is missing. (Notice that the differential equation (15) is called "a ***second-order*** differential equation", because the maximum order of derivatives - of the function $x(t)$ - that appear is 2.) Then we can reduce the order of this differential equation by performing a change of function. The new function will be:

$$v(t)=x'(t) \quad (17)$$

The differential equation in the function $v(t)$, is obtained from (15) and (17):

$$F(t)-k_f v(t)=mv'(t) \quad (18)$$

The equation (18) is a ***first-order*** *nonhomogeneous linear differential equation with constant coefficients.* But for the equation (18), we also have an initial condition:

$$v(0)=0.$$

So we have to solve a Cauchy problem:

$$\begin{cases} mv'+k_f v=F \\ v(0)=0 \end{cases} \quad (19)$$

Next we will solve this Cauchy problem.

Step 1. We start with solving the homogeneous differential linear equation attached to the equation (18):

$$mv'+k_f v=0 \quad (20)$$

The equation (18) is an equation with separable variables. Now we will separate the variables. $mv'=-k_f v\Rightarrow v'=-\frac{k_f}{m}v$. Since $v=v(t)$, it follows:

$$\frac{dv}{dt}=-\frac{k_f}{m}v.$$

In Section 2.2.A, we have shown that in order to separate the variables in this equation, we should multiply with dt and divide by $v\left(=v(t)\right)$. From a mathematical point of view, we should discuss two cases:

Case A) $v\neq 0\left(\text{that is, there exists } t \text{ with } v(t)\neq 0\right)$;

Case B) $v=0\ \left(v(t)=0, \text{ for any } t\right)$.

But, from a physical point of view, Case B) no longer makes sense, because when the car starts moving, its speed is such that $v>0$. So $\frac{dv}{v}=-\frac{k_f}{m}dt$. Integrating, it follows that:

$$\int\frac{1}{v}dv=\int-\frac{k_f}{m}dt\Rightarrow\ln|v|=-\frac{k_f}{m}t+\ln k,$$

where $k>0\Rightarrow$

$$\ln\frac{|v|}{k}=-\frac{k_f}{m}t\Rightarrow|v|=ke^{-\frac{k_f}{m}t}$$

$$v(t)=\pm ke^{-\frac{k_f}{m}t}.$$

We denote $\pm k=C$ and, since $k>0$, it follows that $C\in\mathbb{R}^*$. Then the solution of the *homogeneous* differential equation is:

$$v_o(t)=C\cdot e^{-\frac{k_f}{m}t}, \tag{21}$$

where $C\in\mathbb{R}^*$.

Step 2. Now we are looking for a **particular solution** of the ***nehomogeneous*** differential equation that appears in (19), such that it has the form of v_o from (21), in which, the constant C replaced by a function not known for instant $C(t)$. Hence, by using the ***Method of variation of constant*** (***Euler-Lagrange Method***) we are looking for a particular solution having the form:

$$v_p(t)=C(t)\cdot e^{-\frac{k_f}{m}t}. \tag{22}$$

We "force" $v_p(t)$ from (22), to verify the nonhomogeneous differential equation (19). It follows:

$$mv_p'+k_fv_p=F\Rightarrow m\left(C(t)\cdot e^{-\frac{k_f}{m}t}\right)'+k_f\left(C(t)\cdot e^{-\frac{k_f}{m}t}\right)=F$$

$$\Rightarrow mC'(t)e^{-\frac{k_f}{m}t}+mC(t)e^{-\frac{k_f}{m}t}\left(-\frac{k_f}{m}\right)+k_fC(t)e^{-\frac{k_f}{m}t}=F\Rightarrow$$

$$\Rightarrow m\cdot C'(t)e^{-\frac{k_f}{m}t}=F\left|\cdot\frac{e^{\frac{k_f}{m}t}}{m}\right.\Rightarrow C'(t)=\frac{F\cdot e^{\frac{k_f}{m}t}}{m}\Rightarrow$$

$$\Rightarrow C(t)=\int\frac{F}{m}\cdot e^{\frac{k_f}{m}t}dt. \tag{23}$$

We denote $\frac{k_f}{m}t=u\Rightarrow du=\frac{k_f}{m}dt$. Replacing in (23), it follows:

$$C(t)=\frac{F}{k_f}\int e^{\frac{k_f}{m}t}\cdot\frac{k_f}{m}dt=\frac{F}{k_f}\int e^u du\Rightarrow C(t)=\frac{F}{k_f}\cdot e^u+C_0=\frac{F}{k_f}e^{\frac{k_f}{m}t}+C_0,$$

where $C_0\in\mathbb{R}$. Since from (21), $v_p(t)=C(t)e^{-\frac{k_p}{m}t}$, it follows:

$$v_p(t)=\left(\frac{F}{k_f}\cdot e^{\frac{k_f}{m}t}+C_0\right)e^{-\frac{k_f}{m}t},$$

where $C_0\in\mathbb{R}$, or equivalently

$$v_p(t)=\frac{F}{k_f}+C_0e^{-\frac{k_f}{m}t},$$

with $C_0\in\mathbb{R}$.

Step 3. The **general solution** of the ***nonhomogeneous*** equation is:

$$v_n(t)=v_o(t)+v_p(t)\Rightarrow$$

$$v_n(t)=C\cdot e^{-\frac{k_f}{m}t}+\frac{F}{k_f}+C_0e^{-\frac{k_f}{m}t}$$

$$v_n(t)=(C+C_0)\cdot e^{-\frac{k_f}{m}t}+\frac{F}{k_f}.$$

If we denote $C+C_0=C_1$, it follows:

$$v_n(t)=C_1e^{-\frac{k_f}{m}t}+\frac{F}{k_f}. \tag{24}$$

But $v_n(0)=0,$ from the initial condition (16) of the Cauchy problem that we need to solve $(v(0)=0)$. It follows $C_1+\frac{F}{k_f}=0\Rightarrow C_1=-\frac{F}{k_f}$. We replace this value of the constant C_1 in (24). It follows: $v_n(t)=-\frac{F}{k_f}\cdot e^{-\frac{k_f}{m}t}+\frac{F}{k_f}\Rightarrow v_n(t)=\frac{F}{k_f}\left(1-e^{-\frac{k_f}{m}t}\right)$. So the ***cruising speed*** of the car in the simplifying assumptions imposed at the beginning of our mathematical modelling is:

$$v(t)=\frac{F}{k_f}\left(1-e^{-\frac{k_f}{m}t}\right). \tag{25}$$

Since $v(t)=x'(t)$, from (25) we also get the ***equation of motion*** of the car:

$$x(t)=\int v(t)dt\overset{(25)}{=}\int\frac{F}{k_f}\left(1-e^{-\frac{k_f}{m}t}\right)dt\Rightarrow x(t)=\frac{F}{k_f}\left(t-\right.$$

,

where $C_2\in$.

The relation (26) which follows, gives us the *equation of motion of the car in the matter of cruising speed.*

$$x(t)=\frac{F}{k_f}\left(t+\frac{m}{k_f}e^{-\frac{k_f}{m}t}\right)+C_2 \tag{26}$$

with $C_2\in$.

We are going to determine the integration constant C_2. We use the *initial condition* (16) $(x(0)=0)$. We replace $t=0$ in (26), and it follows:

$$0=x(0)=\frac{F}{k_f}\cdot\frac{m}{k_f}+C_2\Rightarrow C_2=-\frac{mF}{k_f^2}.$$

Replacing in (26), this value of the integration constant C_2, it follows:

$$x(t)=\frac{F}{k_f}\left(t+\frac{m}{k_f}e^{-\frac{k_f}{m}t}-\frac{m}{k_f}\right).$$

Ω

5. Discussions

The idea of this paper is not new to its authors. Indeed, this paper is added to the following works (among their authors being two of the authors of the current work) : [Dăneţ, Dilimoţ & Popescu, 2009], [Dăneţ, Popescu & Dilimoţ, 2008], [Dăneţ, Popescu & Dilimoţ, 2010], [Dăneţ et.al, 2014], [Dăneţ, Popescu & Voicu, 2008], [Dăneţ, Popescu & Voicu, 2009], [Dăneţ, Popescu & Voicu, 2011] and [Dăneţ, Popescu & Voicu, 2010]

In all these works, we set out to change the face of teaching mathematics in a technical university, especially for first-year students who are preparing to become civil engineers. The main idea is that the rigorous, technical, specific exposition of mathematics teaching in a mathematics college is complemented by an informal exposition. Thus we can add historical information, we can find the motivation of the discussed topic, we can present some applications mainly in Physics or in engineering (or, sometimes, in economics) and we can formulate problems that remained open. In this way, the exposure becomes more accessible.

To these we add something that comes from our rich teaching experience as teachers in the Department of Mathematics of a technical university.

First we started from the hypothesis that the students to whom the courses are dedicated, possess certain calculation algorithms, but have no experience or have only a limited one, in the technique of rigorous demonstrations.

Therefore, the exposure is built gradually, with great care in choosing the titles corresponding to the different moments of the course. This is because we consider that, at an early stage, the knowledge of these titles is a first phase of awareness of the topics addressed, being a first summary of them.

The gradual exposition, step by step, follows the theoretical construction of the courses, but it is also modelled according to the students' capacity to assimilate this construction.

Thus the structure of the course contains the following steps:

- First of all the fundamental notions, which constitute the dictionary of the respective subject, as well as examples meant to facilitate the understanding of these notions;
- Then, the results that aim to fix the fundamental properties of the notions in the "dictionary", the connections between these notions, as well as certain calculation algorithms;
- It follows some proofs of these results, namely those that handle only the notions that appear and have the role of "theoretical exercises";
- Continues with examples of application of these results and with solved exercises, exploring the classic types of applications, as "models" of using the "theory" previously exposed;
- As a recapitulation of the main notions and results and of the fundamental algorithms, for each lesson a questionnaire is formulated that summarizes, with as few questions as possible, the previous "steps". The answers to these "questionnaires", optionally offered by students who want to be better prepared for the exam, are a summary of the course, so "course notes" made individually. (Note that these questionnaires are distributed to students at the beginning of each semester);
- Some proposed exercises.

Each step assimilated by the student is a step towards the maximum grade. In order to obtain the credits necessary to pass the exam, the students can only know the dictionary and some suitable examples as well as the fundamental algorithms. This *step-by-step approach* provides accessibility to both students who do not have a solid foundation and those who have such a foundation. It is a modern, original and attractive approach, presenting in a new way a standard and classic material. This smooths the transition from a more rigorous approach to one more accessible.

6. Conclusions

This paper, together with the papers mentioned in the previous section, serve the purpose for which they were written. As mentioned earlier, the goal is twofold.

On the one hand, all these works offer students a means of approaching some topics that they will study later, in other courses. On the other hand, the approach in our papers illustrates, as suggestively as possible, that what is taught in mathematics courses will be useful to students in understanding other disciplines, helping them to *rigorously* solve problems specific to these disciplines. We emphasize that, here, by "*rigorously* resolving" we mean "giving a solution, in which each statement is the clear consequence of the statements previously demonstrated". It is therefore a matter of cultivating logic and correctness of reasoning and in a technical university, this mission belongs to mathematics courses.

Authors note:

Ştefania Constantinescu is a PhD University Assistant for over 10 years at the Technical University of Civil Engineering Bucharest (Department of Mathematics and Computer Science). She is the author of some books and collections of problems for students and of some works in Applied Mathematics. Areas of competence: Applied Functional Analysis and Applied Complex Analysis (Signals Theory) and Measure Theory.

Rodica-Mihaela Dăneţ is a PhD University Professor, had working in the Department of Mathematics and Computer Science, and now working in the Research Center for Mathematics and Computer Science at the Technical University of Civil Engineering Bucharest, Graduate of the Faculty of Mathematics-Mechanics, University of Bucharest. PhD in Mathematics (in Functional Analysis). She has published many papers, and led teams of young researchers in the elaboration of some scientific and didactic works, presented at national and international scientific events. Centers of interest and areas of professional competence are: theory of topological ordered vector spaces and linear operators, extension problems for positive linear operators, interval analysis, fixed-point theory and coincidence point theory for multimaps, general equilibrium theory, applications of this theory in economics, teaching mathematics in engineering education, mathematics in physics of buildings, didactical works for pre-university education.

Marian-Valentin Popescu is a PhD University Lecturer, working in the Technical University of Civil Engineering Bucharest, the Department of Metal Construction, Management and Engineering Graphics, and the Research Center for Mathematics and Computer Science, in close cooperation with the Department of Mathematics and Computer Science. Graduate of the Faculty of Mathematics, University of Bucharest. PhD in Civil Engineering and Installations. He has published scientific and didactic papers in the mathematics domain, presented at national and international scientific events.

References

Amontons, G. (1699). De la resistance causée dans les machines (About resistance and force in machines). *Mémoires de l'Académie Royale*, A, Chez Gerard Kuyper, Amsterdam, published in 1706, 257–282.

Braun, O.; Peyrard, M. (2011). Dependence of kinetic friction on velocity: Master equation approach. Physical Review E: Statistical, Nonlinear and Soft Matter Physics. *American Physical Society*, *83*, 046 129. 1-9.

Dăneț, R-M.; Dilimoț-Niță, S-V.; Popescu, M-V. (2009). Sequences and series applied in the physics of buildings. *Mathematical Modelling in Civil Engineering*, *vol. 5*, No. 3, (Scientific Journal of the Technical University of Civil Engineering Bucharest), 36-50.

Dăneț, R-M.; Popescu, M-V.; Dilimoț-Niță, S-V. (2008). Mathematics in physics of buildings (1): Harmonic series, Euler's Constant, Recurrent sequences. The XVth Conference with international participation "Comfort, Efficiency, Energy Conservation and Environmental Protection" (CECEPM-XV), November 26-27, 2008, Technical University of Civil Engineering Bucharest, Faculty of Building Services, *Proceedings of the Conference*-electronic issue [ISSN 1842-6131].

Dăneț, R-M.; Popescu, M-V.; Dilimoţ-Niţă, S-V. (2010). Mathematics in physics of buildings (2), n -dimensional spaces, composed mappings, vector functions of several variables and their extreme points, linear functions and their matrix representation, optimization methods applied in thermodynamics. The XVIth Conference with international participation "Comfort, Efficiency, Energy Conservation and Environmental Protection" (CECEPM-XVI), March 18-19, 2010, Technical University of Civil Engineering of Bucharest, Faculty of Building Services, *Proceeding of the Conference*, electronic issue, [ISSN 1842-6131].

Dăneţ, R-M.; Popescu, M-V.; Popescu, I.; Popescu, N. (2014). A new approach to teach mathematics for engineers (1). Numerical sets. *Romanian Journal of Civil Engineering*, *vol.5*, No. 2, Bucharest: Ed. Matrix Rom, 126-157.

Dăneț, R-M.; Popescu, M-V.; Voicu, F. (2009). On a quasi-variational inequality among some problems with engineering applications. *Mathematical Modelling in Civil Engineering*, *vol.5*, No. 7, (Scientific Journal of the Technical University of Civil Engineering Bucharest), 52-67.

Dăneț, R.- M.; Popescu, M-V.; Voicu, F. (2011). A new approach to teaching mathematics in engineering education. *Mathematical Modelling in Civil Engineering*, *vol. 7*, No. 3, (Scientific Journal of the Technical University of Civil Engineering Bucharest), 33-55.

Dăneț, R-M.; Popescu, M-V.; Voicu, F. (2008). A new result on variational inequalities, an applied subject in engineering. The XVth Conference with international participation "Comfort, Efficiency, Energy Conservation and Environmental Protection" (CECEPM-XV), November 26-27, 2008, Technical University of Civil Engineering Bucharest, Faculty of Building Services, *Proceedings of the Conference*-electronic issue [ISSN 1842-6131].

Dăneț, R-M.; Popescu, M-V. ; Voicu, F. (2010). How we can use the differential calculus to compute the demand in an arbitrary market. The XVIth Conference with international participation "Comfort, Efficiency, Energy Conservation and Environmental Protection" (CECEPM-XVI), March 18-19, 2010, Technical University of Civil Engineering of Bucharest, Faculty of Building Services, *Proceeding of the Conference*, electronic issue, [ISSN 1842-6131].

Dowson, D. (1979). *History of Tribology*. London: Longman.

Popov, L.V. (2010). *Coulomb's law of Friction in Contact Mechanics and Frictions, Physical Principles and Applications*. Berlin, Heidelberg: Springer, 133-154.

Popova, E.; Popov, V.L. (2015). The research works of Coulomb and Amontons and the generalized laws of friction. *Friction 3* (2), 183-190.

Răduleț, R. et. al (1957-1958). *Lexiconul Tehnic Român*. București: Editura Tehnică.

Tamm, I. (1952). *Bazele teoriei electricității*. București: Editura Tehnică.